§13. p'-Automorphisms of p-groups.

(A) Let $V = X \oplus Y$ be an additive Abelian p-group and let Γ be a p'-group operating on V and leaving the direct summand X invariant: so $v(\xi\eta) = (v\xi)\eta$ for $v \in V$, ξ and η in Γ; $(u+v)\xi = u\xi + v\xi$ for $u, v \in V$, ξ in Γ; and $X\Gamma = X$.

Every ~~.....~~ $v \in V$ is uniquely expressible in the form $x + y$ with $x \in X$, $y \in Y$. For $y \in Y$, $\xi \in \Gamma$ we can therefore define $y^\xi \in Y$ and $y_\xi \in X$ by the equation

$$y\xi = y^\xi + y_\xi .$$

Expressing that $y(\xi\eta) = (y\xi)\eta$, we then obtain

$$y^{\xi\eta} = (y^\xi)^\eta \quad \text{and} \quad y_{\xi\eta} = {y^\xi}_\eta + y_\xi\eta \tag{1}$$

for all $\xi, \eta \in \Gamma$.

Let $|\Gamma| = g$. Since $(g, p) = 1$, we can choose an integer h such that $gh \equiv 1 \mod |V|$ and so $ghv = v$ for all $v \in V$. Define

$$w(y) = h \sum_{\eta \in \Gamma} {y^\eta}_{\eta^{-1}} , \tag{2}$$

so that $y \to w(y)$ is a mapping of Y into X. Then for $\xi \in \Gamma$,

$$w(y^\xi) = h \sum_\eta {y^{\xi\eta}}_{\eta^{-1}} = h \sum_\eta {y^\eta}_{\eta^{-1}\xi}$$

by the first equation of (1) and so, by the second equation of (1), we h[ave]

$$w(y^\xi) = h \sum_\eta (y_\xi - y_\eta \eta^{-1}\xi) = y_\xi - \left(h \sum_\eta y_\eta \eta^{-1}\right)\xi .$$

But $y1 = y$ and so $y_1 = 0$ if 1 is the unit element of Γ. So (1) gi[ves] $y_\eta \eta^{-1} = -{y^\eta}_{\eta^{-1}}$ and we obtain

$$w(y^\xi) = y_\xi + w(y)\xi \tag{3}$$

For y_1, y_2 in Y we have $(y_1 + y_2)\xi = y_1\xi + y_2\xi$ and hence $(y_1+y_2)^\xi = y_1^\xi + y_2^\xi$, $(y_1+y_2)_\xi = (y_1)_\xi + (y_2)_\xi$ and so

$$w(y_1 + y_2) = w(y_1) + w(y_2) \tag{4}.$$

So $y \to w(y)$ is a homomorphism of Y into X and the set T of all

$$t(y) = y + w(y) \qquad (y \in Y)$$

is therefore a subgroup of V complementary to X, i.e. $V = X \oplus T$.

omission Since $t(y)\xi = y\xi + w(y)\xi = y^\xi + y_\xi + w(y)\xi = y^\xi + w(y^\xi) = t(y^\xi)$ by (3), we see that T is Γ-invariant. This gives

<u>Theorem 13·1</u> Let the p'-group Γ be represented by automorphisms of the additive Abelian p-group V. If Γ leaves invariant a direct summand X of V, then there is a Γ-invariant subgroup T such that $V = X \oplus T$.

Note that the number of subgroups Y of V such that $V = X \oplus Y$ is equal to $|\mathrm{Hom}(V/X, X)|$, which is a power of p. Hence 13·1 is immediate in the case $|\Gamma| = q^m$, q prime.

V is called <u>Γ-indecomposable</u> if it cannot be expressed as the direct sum of two proper Γ-invariant subgroups. Obviously, we can always express V as a direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_r$ $(r \geqslant 0)$ of ~~indecomp~~ Γ-invariant subgroups $V_i \neq 0$ each of which is Γ-indecomposable. The V_i a <u>Γ-indecomposable components</u> of V.

An Abelian p-group is called <u>homocyclic</u> if all its invariants are equa

~~Lemma~~ <u>Corollary 13·11</u>. The Γ-indecomposable components of V are all homocyclic

Proof: We may assume that V is Γ-indecomposable. Suppose if possible that V is not homocyclic and let l be the largest invariant of V. Then $l > 1$ and if $W = \Omega_1 V$ and $X = W \cap \mho_{l-1} V$, we have $0 < X < W$. Since W is elementary, X is a direct summand of W. Since X and W are characteristic in V, they are Γ-invariant. Hence there is a Γ-invarian subgroup T_1 such that $W = X \oplus T_1$ by 13·1. Since $X \cap T_1 = 0$, the largest invariant of $V_1 = V/T_1$ is still l. If V_1 is not homocyclic, we can proceed similarly to the existence of a Γ-invariant subgroup $T_2 > T_1$ such that $T_2 \cap X + T_1 = T_1$ and so $T_2 \cap X = 0$, and the largest invariant of $V_2 = V/T_2$ is therefore still l. Continuing in this way, we eventually find a Γ-invariant subgroup T of V such that $X \cap T = 0$ and V/T is homocyclic. If $|X| = p^r$, this implies that V/T is of type $\underbrace{(l, l, \dots, l)}_{r}$ Here r is the multiplicity of l as an invariant of V, and V contains subgroups $U \cong V/T$. For any such U, $\Omega_1(U) = \mho_{l-1}(U) = X$ and so $V = T \oplus$

It follows from 13·1 that U may be chosen to be Γ-invariant. This contradicts the Γ-indecomposability of V and we conclude that V must in fact be homocyclic.

(B) <u>Theorem 13·2</u> Let the p'-group Γ be represented by automorphisms of the p-group G, let $H = [G, \Gamma]$ and let $C = C_G(\Gamma)$. Then (i) $G = HC$; (ii) $[H, \Gamma] = H$ and (iii) if $H \leq \varphi(G)$, then $H = 1$.

Here C is the subgroup of all $\xi \in G$ which are left invariant by every $\gamma \in \Gamma$. We can form the split extension $G\Gamma$ of G by Γ determined by the given representation of Γ. By 7·1 (iii), $H \lhd G\Gamma$.

Proof: (iii). Let $\Phi = \varphi(G)$. By 5·2 (vi), G/Φ is an elementary Abelian p-group. Let $|G : \Phi| = p^r$ and let $\Phi\xi_1, \cdots, \Phi\xi_r$ be a basis of G/Φ. By 9·1 (i), $G = \{\xi_1, \cdots, \xi_r\}$. Let A_1 be the group of all automorphisms $\alpha \in \mathrm{Aut}\, G$ such that $[G, \alpha] \leq \Phi$. Thus $A_1 \lhd A = \mathrm{Aut}\, G$ and A/A_1 is isomorphic with a subgroup of $\mathrm{Aut}\, G/\Phi$, which has order $(p^r-1)(p^r-p)\cdots(p^r-p^{r-1})$ by 8·81. If $\alpha \in A_1$, then $[\xi_i, \alpha] \in \Phi$ for each i; and since the ξ_i generate G, α is uniquely determined by the r elements $[\xi_i, \alpha]$. More generally, if $\xi_i\eta_i^{-1} \in \Phi$ for each i, then $G = \{\eta_1, \cdots, \eta_r\}$ and so no element $\alpha \neq 1$ in A_1 leaves any such ordered set $(\eta_1, \cdots, \eta_r)$ invariant. The total number of such ordered sets is $|\Phi|^r$. Hence $|A_1|$ divides $|\Phi|^r$ and thus A_1 is a p-group. If $H \leq \Phi$, the automorphisms of G which represent Γ all belong to A_1 and since Γ is by hypothesis a p'-group, it follows that $H = 1$ and $C = G$. Thus (iii) is proved.

The principle used here may be worth stating as

<u>Lemma 13·21</u> Let G be any group and K any ~~normal~~ characteristic subgroup of G contained in $\varphi(G)$. Let $A = \mathrm{Aut}\, G$ and let A_1 consist of all $\alpha \in A$ such that $[G, \alpha] \leq K$, so that $A_1 \lhd A$ and A/A_1 is isomorphic with the subgroup of $\mathrm{Aut}\, G/K$ induced by elements of A. Then $|A_1|$ divides $|K|^r$, where r is the minimum number of generators of G.

Note that r is also the minimum number of generators of G/K and equally of $G/\varphi(G)$.

(i) Suppose first that $H \leq \zeta G$. Then the mapping $\xi \rightarrow [\xi, \gamma]$, $(\xi \in G)$, is homomorphic by 7.1 (i) for each $\gamma \in \Gamma$. Since $[G, \gamma] \leq H$, which is Abelian, the kernel of this homomorphism contains G', by 7.1(vii). This is true for each $\gamma \in \Gamma$. Hence $G' \leq C$. G/G' is a normal Abelian Sylow p-subgro of $G\Gamma/G'$ and so, by 12.5, it is the direct product of $\bar{C}/G'$ and $\bar{H}/G'$ where $\bar{H} = HG'$ and $\bar{C}$ consists of all $\xi \in G$ such that $[\xi, \Gamma] \leq G'$. Henc $C \leq \bar{C}$. If $\bar{H} = G'$, we have $H = 1$ by (iii) and $C = G$ and the result follows. If $\bar{H} > G'$, then $\bar{C} < G$ and we may assume by induction on $|G|$ that $\bar{C} = [\bar{C}, \Gamma]C$. But $[\bar{C}, \Gamma] \leq \bar{C} \cap \bar{H} = G' \leq C$ and so $\bar{C} = C$. Since $H \lhd G$, CH is a subgroup of G and we have $CHG' = \bar{C}\bar{H} = G$. Henc

should be G

$CH = G'$ by 9.1 (i) and 5.2 (vi).

If $H \nleq \zeta G$, let $K = H \cap \zeta G$. Then $1 < K \lhd G\Gamma$. Let C_1 consist of all $\xi \in G$ such that $[\xi, \Gamma] \leq K$. Then $C \leq C_1$ and by induction we may assume that $HC_1 = G$. Since $H \nleq \zeta G$, we have $C_1 < G$. By induction again we may assume that $[C_1, \Gamma]C = C_1$. This gives $HC = H[C_1, \Gamma]C = HC_1$ since $[C_1, \Gamma] \leq H$. Thus (i) is proved.

(ii) Let $H_1 = [H, \Gamma]$. By (i), $H \leq H_1C$ and so $G = H_1C$. ~~Every element of G has the form $\eta_1\xi$ with $\eta_1 \in H_1$ and $\xi \in C$. By 7.1(iii), Γ normalizes~~ Hence $G = CH_1$ and every element of G has the form $\xi\eta_1$ with $\xi \in C$, $\eta_1 \in H_1$. Therefore H is generated by the elements $[\xi\eta_1, \gamma] = [\eta_1, \gamma]$, with $\eta_1 \in H_1$ and $\gamma \in \Gamma$. But H and H_1 are normalized by Γ, so $[\eta_1, \gamma] \in H_1$ and $H_1 \leq H$. Hence $H_1 = H$.

(C) ~~4~~. <u>Lemma 13·3</u> Let G be a p-soluble group, let M be a minimal normal subgroup of G and suppose that $|M| = p^m$ with $m > 1$, but that every chief p-factor of G/M has order p. Then G is the split extension of M by a maximal subgroup H of index p^m in G.

Proof: If $K \lhd G$ and $K \cap M = 1$, then G/K satisfies the conditions prescribed for G. If H/K is a subgroup transversal to KM/K in G/K, then $H \cap M = 1$ and $HM = G$. By induction on $|G|$, we may therefore assume that M is the only minimal normal subgroup of G.

Let L/M be a chief factor of G, and suppose first that L/M is a p'-group. By 9·4, L has a single class of conjugate $S_{p'}$-subgroups H and these are intravariant in L. If $N = N_G(H)$, it follows that $NL = G$ by 10·8 (iii). Since $HM = L$, we have $NM = G$. Since M is Abelian and normal in G, $N \cap M$ is normal in $NM = G$. Since M is a minimal normal subgroup of G, either $N \cap M = 1$ or $M \leq N$. In the second case, $H \lhd L$ and, as a normal $S_{p'}$-subgroup of L, H is even characteristic in L. This would imply $H \lhd G$ contrary to the assumption that M is the only minimal normal subgroup of G. Hence $N \cap M = 1$ and the theorem follows.

Since G is p-soluble, L/M if not a p'-group must be a p-group and therefore, by the assumption about G, we have $|L:M| = p$. Then L is a p-group and $\mathfrak{z}L \cap M \neq 1$ by 5·2 (i). Since $\mathfrak{z}L \cap M \lhd G$ it follows that $M \leq \mathfrak{z}L$. But L/M is cyclic and hence $L = \mathfrak{z}L$ is Abelian by 7·2 (iii). Since M is elementary of order p^m, L must be elementary of order p^{m+1}; for the only alternative is for L to be of type $(1^{m-1}2)$ and since $m > 1$ this would imply $M = \Omega_1(L) > \mho_1(L) > 1$, $\mho_1(L) \lhd G$, contrary to the hypothesis that M is a minimal normal subgroup of G.

Now let $C = C_G(L)$. Then $C < G$. Let D/C be a chief factor of G.

Suppose first that D/C is a p-group. Then $|D:C| = p$ by hypothesis and $C_M(D) \neq 1$ by 5.1. Since $C_M(D) \lhd G$, it follows that $[M, D] = 1$. Let $D = \{C, \xi\}$ and $L = \{M, \eta\}$. Since $[L, C] = 1$, $[L, D]$ is generated by $\zeta = [\xi, \eta]$ and its conjugates in D. But $[L, D] \leq M$ by 5.1 and so $[\zeta, D] = 1$. Hence $[L, D] = \{\zeta\}$ is cyclic. Since $[L, D] \lhd G$, this contradicts the hypothesis that M is a minimal normal subgroup of G.

We conclude that D/C must be a p'-group Γ. This is represented by automorphisms of the elementary Abelian p-group L. By 13.1, we have L as the direct product $L_0 L_1 \cdots L_r$ of a certain number of minimal normal subgroups L_i of D, where we may assume that $M = L_1 L_2 \cdots L_r$, so that $|L_0| = p$. Joining together those L_i which are Γ-isomorphic to a given one, we obtain L as the direct product $W_0 W_1 \cdots W_s$, where each W_j is a Wedderburn component of L with respect to Γ. Let W_0 contain L_0. Then $W_1 W_2 \cdots W_s \leq M$. Since $D \lhd G$, the mapping $W_i \to W_i^{\xi}$ is a permutation of $W_0, \ldots, W_s$, as in theorem 8.9. Since $M \lhd G$, it follows that $W_0 \lhd G$ and since M is minimal normal, we must have $W_0 \cap M = 1$ and so $W_0 = L_0$. This contradicts the hypothesis that M is the only minimal normal subgroup of G. So this case also is impossible and we conclude that L/M has to be a p'-group.

This concludes the proof of 13.3.

<u>Theorem 13.4</u> Let M be a maximal subgroup of G.

(i) If $|G:M|$ is either a prime or the square of a prime for all M, then G is soluble.

(ii) If $|G:M|$ is a prime for all M, then G is supersoluble.

Note that (ii) is the converse of part of 12.9(ii). It allows us to characterize supersoluble groups as those groups in which every maximal subgroup is of prime index. This result is due to B. Huppert.

Proof: (i). Let p be the largest prime divisor of $|G|$, let P be a Sylow p-subgroup of G. If $P \lhd G$, we may assume that G/P is soluble by induction on $|G|$. Hence G is soluble, since P is a p-group and therefore certainly soluble. If $N = N_G(P) < G$, let M be a maximal subgroup of G containing N. Then $|G:M| = q$ or q^2 for some prime $q < p$, by hypothesis. But $|G:M| \equiv 1 \bmod p$ by 5·43. Since $(p, q-1) = 1$, we must have $|G:M|$ and p divides $q+1$. Hence $p = 3$, $q = 2$ and G is soluble by 9·8, corollary.

(ii) By (i), G is soluble. Let $G = G_0 > G_1 > \cdots > G_m = 1$ be a chief series of G. If G is not supersoluble, let i be the least integer for which $|G_{i-1} : G_i|$ is not a prime. Then G_{i-1}/G_i is an elementary Abelian p group of order p^m, $m > 1$, for some prime p; and the group G/G_i satisfies the hypotheses of 13·3 with $G_{i-1}/G_i = M$. Hence there is a maximal subgroup H of G such that $H \cap G_{i-1} = G_i$ and $HG_{i-1} = G$. Then $|G:H| = p^m$, contrary to the assumption that all maximal subgroups of G are of prime index. We conclude that G is supersoluble.

(D) <u>Theorem 13·5</u> If every maximal subgroup of G is nilpotent, then G is soluble.

This theorem is due to Otto Schmidt and was found later independently by Iwasawa.

Proof: Suppose there are two distinct maximal subgroups M_1 and M_2 of G such that $D = M_1 \cap M_2 \neq 1$. Choose M_1 and M_2 so that D is as large as possible, and let $N = N_G(D)$. By 6·8 (iii), $D < M_1 \cap N = N_{M_1}(D)$ and similarly $D < M_2 \cap N$. If $N < G$, let M be a maximal subgroup of G containing N. By the maximality of D, we should then have $M_1 = M = M_2$ contrary to $M_1 \neq M_2$. Hence $N = G$ and $D \triangleleft G$. By induction on $|G|$ we may assume that G/D is soluble. Since D is nilpotent, it follows that G is soluble.

We may now assume that $M_1 \cap M_2 = 1$ for ~~any~~ every pair of maximal subgroups M_1 and M_2 of G. If $M_1 \triangleleft G$, we obtain the solubility of G by induction on $|G|$, since $M_1 = 1$ implies that G is cyclic of order a prime and in any case M_1 is supposed to be nilpotent. We may therefore suppose that no maximal subgroup M_1 of G is normal in G. Let $|M_1| = m_1$ and $|G : M_1| = n_1$. Then M_1 has n_1 conjugates in G and together these contain exactly $1 + n_1(m_1 - 1)$ elements of G. This number is less than $|G| = n_1 m_1$ and so G must have a maximal subgroup M_2, of order m_2 and index n_2 in G, which is not conjugate to M_1 in G. Since M_2 and its conjugates contain besides the unit element exactly $n_2(m_2 - 1)$ further elements of G, we obtain $|G| = n_2 m_2 \geqslant 1 + n_1(m_1 - 1) + n_2(m_2 - 1)$ and so $n_2 - 1 \geqslant n_1(m_1 - 1) \geqslant n_1$. Similarly, $n_1 - 1 \geqslant n_2$, which is a contradiction. Thus 13.5 is proved

(E). Suppose that G is not nilpotent but that all proper subgroups of G are nilpotent. By 13.5, G is soluble. Hence G has a maximal normal subgroup M of index a prime p and so $G = \{M, \xi\}$ where ξ is of order p^r for some r and $\xi^p \in M$. Since M is nilpotent but not G, we have $M = \mathfrak{n}G$ the Fitting subgroup of G. For some $q \neq p$, the Sylow q-subgroup Q of M must contain an element η which does not commute with ξ; for otherwise G would be nilpotent. Then $G = \{\xi, \eta\}$ since $\{\xi, \eta\}$ is not nilpotent; and so $Q = \{\eta^G\}$ and M is the direct product of Q with $\{\xi^p\}$. Thus G is the split extension of Q by the cyclic p-group $\{\xi\}$. Moreover every proper subgroup Q_1 of Q which is normalized by ξ must be centralized by ξ; for otherwise $\{Q_1, \xi\} = G_1$ would be a proper subgroup of G but not nilpotent.

We consider a slightly more general situation in

Theorem 13.6 Let the q'-group Γ be represented by automorphisms of the q-group Q and suppose that there is an element $\gamma \in \Gamma$ which centralizes every proper subgroup of Q which is normal in the split extension $Q\Gamma$ but which does not centralize Q. Then $Q' \leqq \mathfrak{z}Q$ and Q/Q' is a chief factor of $Q\Gamma$. Either Q is elementary; or Q is of class 2 with Q/Q' and Q' both elementary and $Q' = \mathfrak{z}Q$.

Proof: Since $Q' \leq \varphi(Q)$ and γ has order prime to q and does not centralize Q, it follows from 13.21 that γ does not centralize Q/Q'.

Hence Q/Q' is Γ-indecomposable; for otherwise there would be a Γ-invariant subgroup Q_1 containing Q' and such that $[Q_1, \gamma] \not\leq Q'$, contrary to hypothesis. By 12·5, Q/Q' is the direct product of $C_{Q/Q'}(\gamma)$ with another group. Since γ does not centralize Q/Q', it follows that γ does not centralize $\Omega_1(Q/Q')$ either. But $\Omega_1(Q/Q') = Q_2/Q'$ with Q_2 characteristic in Q. Hence $Q = Q_2$ and Q/Q' is elementary. Since it is Γ-indecomposable, it is therefore a chief factor of $Q\Gamma$.

Let $\Gamma_0 = \{\gamma^\Gamma\}$. Then $\Gamma_0 \triangleleft \Gamma$; and Γ_0 centralizes Q' because γ does so. A $[Q, \Gamma_0] \triangleleft Q\Gamma$ and $[Q, \Gamma_0]$ is not contained in Q' since γ does not centraliz Q/Q'. Since the latter is a chief factor of $Q\Gamma$, it follows that $Q'[Q, \Gamma_0] = Q$ and hence $[Q, \Gamma_0] = Q$ by 9·1 (i). Since $[Q', \Gamma_0] = 1$, both $[Q', Q, \Gamma_0]$ and $[\Gamma_0, Q', Q]$ are equal to 1. Hence $[Q, Q'] = [Q, \Gamma_0, Q'] =$ by 7·7 (ii). Thus $Q' \leq \mathfrak{z}Q$.

If Q is not elementary, we have $Q' \neq 1$ and so $Q' \leq \mathfrak{z}Q < Q$. But $\mathfrak{z}Q \triangleleft Q\Gamma$ and Q/Q' is a chief factor of $Q\Gamma$; so in this case $Q' = \mathfrak{z}Q$. Let ξ, η be in Q and let $\zeta = [\xi, \eta]$. Then $\zeta \in \mathfrak{z}Q$ and $\eta^{-1}\xi\eta = \xi\zeta$ and so $\eta^{-q}\xi\eta^{q} = \xi\zeta^{q}$. But $\eta^q \in Q' = \mathfrak{z}Q$ and so η^q commutes with ξ. Hence $\zeta^q = 1$. The Abelian group Q' is generated by elements ζ of order q. Thus Q' is elementary and 13·6 is proved.

~~Lemma 13·52 Let Q be a q-group such that $Q' = \mathfrak{z}Q$ is of order q a prime. Then Q is the central product of a certain number r of non-Abelian groups of order q^3. For each prime q and each integer $r = 1, 2, \dots$ there are exactly two non-isomorphic groups Q of this kind and Q/Q' is elementary of order q^{2r}.~~

~~Proof: Let ξ, η be in Q and let $[\xi, \eta] = \zeta$. By hypothesis $\zeta^q = 1$. As we have just seen, $\eta^{q}\xi\eta^{q} = \xi\zeta^{q}$ since $\zeta \in \mathfrak{z}Q$. Hence η^q commutes with ξ. This is true for all ξ, η in Q. Hence $\eta^q \in Q' = \mathfrak{z}Q$ and Q/Q' is elementary.~~

~~Let $|Q : Q'| = q^s$. Then $s > 1$, since $Q/\mathfrak{z}Q$ cannot be cyclic. Since $Q' \neq 1$, we can choose ξ_1, η_1 in Q such that $\zeta = [\xi_1, \eta_1] \neq 1$.~~

(F) Let Γ be any group, let X be a cyclic group of order p and let $G = X \wr \Gamma$. If $|\Gamma| = n$, the base group V of G is elementary Abelian of order p^n, with a basis x_α $(\alpha \in \Gamma)$ such that $x_\alpha{}^\beta = x_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$. Thus Γ is represented faithfully by automorphisms of V and this is called the <u>regular representation of Γ mod p</u>.

Let $1 = V_0 < V_1 < \cdots < V_r = V$ be part of a chief series of V. Then Γ is represented, with kernel Γ_i say, by automorphisms of V_i/V_{i-1}. If $\Delta = \bigcap_{i=1}^{r} \Gamma_i$, we have $[V_i, \Delta] \leq V_{i-1}$ for all i and so Δ is a p-group by 7·9 (i).

A group Γ is called <u>monolithic</u> if it has only one minimal normal subgroup M. This implies $\Gamma \neq 1$.

<u>Theorem 13·7</u> (i) Let Γ be a monolithic group such that $O_p\Gamma = 1$. Then Γ has a faithful irreducible representation f mod p.

(ii) If $p_1, p_2, \ldots, p_n$ are primes such that $p_{i-1} \neq p_i$ $(i = 2, \ldots, n)$, then there exist groups G with one and only one chief series

$$G = G_0 > G_1 > \cdots > G_n = 1$$

and such that G_{i-1}/G_i is an elementary Abelian p_i-group for each $i = 1, \ldots, n$.

Proof: (i) Since $O_p\Gamma = 1$, we have $\Delta = 1$; and this implies that $\Gamma_i = 1$ for some i, since Γ is monolithic. Hence Γ is represented faithfully and irreducibly by automorphisms of V_i/V_{i-1}.

(ii) When $n = 1$, we can take G to be cyclic of order p_1. Let $n > 1$. By induction we may assume the existence of a group Γ with only one chief series $\Gamma = \Gamma_0 > \Gamma_1 > \cdots > \Gamma_{n-1} = 1$ and such that Γ_{i-1}/Γ_i is a p_i-group for $i = 1, \ldots, n-1$. Then Γ is monolithic. Its unique minimal normal subgroup Γ_{n-2} is a p_{n-1}-group. Since $p_{n-1} \neq p_n$, we have $O_{p_n}\Gamma = 1$. By (i), Γ has a faithful irreducible representation f by automorphisms of an elementary Abelian p_n-group G_{n-1}. Let $G = \langle \Gamma, G_{n-1}; f \rangle$ be the corresponding split extension. Since f is irreducible, G_{n-1} is a minimal normal subgroup of G; it is the only one by 6·9, since f is faithful. Hence G is monolithic. Since $G/G_{n-1} \cong \Gamma$, G has only one chief series, with terms $G_n = 1$, $G_{n-k} = G_{n-1}\Gamma_{n-k}$ $(k = 1, 2, \ldots, n)$ and $G_{i-1}/G_i \cong \Gamma_{i-1}/\Gamma_i$ is a p_i-group, as required.