§9. Frattini subgroups. Split extensions. ϖ-soluble groups. S_ϖ-subgroups

(A). We consider next how to construct larger groups from smaller ones. Let $K \triangleleft G$ and suppose that $1 < K < G$. If there exists in G a proper subgroup H such that $KH = G$, we say that K is partially complemented in G. If in addition $K \cap H = 1$, so that H is transversal to K in G, we say that K is complemented in G.

The intersection of the maximal subgroups of G is called the Frattini subgroup of G and denoted by $\varphi(G)$. Clearly $\varphi(G)$ char G.

Theorem 9.1 Let $F = \mathfrak{N}G$ and $\Phi = \varphi(G)$ be respectively the Fitting and the Frattini subgroups of G. Then

(i) $G = \{\xi_1, \dots, \xi_n\}$ if and only if $G = \{\Phi, \xi_1, \dots, \xi_n\}$.

(ii) Let $K \triangleleft G$. Then K is partially complemented in G if and only if $K \not\leq \Phi$.

(iii) $F' \leq \Phi \leq F$, so that Φ is nilpotent and F/Φ is Abelian. Moreover, $F/\Phi = \mathfrak{N}(G/\Phi)$ the Fitting subgroup of G/Φ.

Proof: (i) Let $H = \{\xi_1, \dots, \xi_n\} < G$. Then there exists a maximal subgroup M of G such that $H \leq M$. Since $\Phi \leq M$, it follows that $\{\Phi, \xi_1, \dots, \xi_n\} \leq M < G$.

(ii) Let H be a proper subgroup of G such that $KH = G$. If $K \leq \Phi$, we should have $\Phi H = G$ and so $H = G$ by (i), contrary to $H < G$. Hence $K \not\leq \Phi$. Conversely, suppose $K \not\leq \Phi$. Then there is a maximal subgroup M of G which does not contain K. Since $K \triangleleft G$, it follows that KM is a group with M as a proper subgroup, hence $KM = G$ and K is partially complemented in G.

(iii) Let $K = \mathfrak{N}(G \bmod \Phi)$ so that K/Φ is the Fitting subgroup of G/Φ. Let S be a Sylow p-subgroup of K. Since K/Φ is nilpotent, $\Phi S/\Phi$ is a characteristic subgroup of K/Φ. Also $K \triangleleft G$, $\Phi \triangleleft G$. Hence $\Phi S \triangleleft G$. As a Sylow subgroup of a normal subgroup of G, S is pronormal in G and so, if $N = N_G(S)$, we have $\Phi N = \Phi S N = G$ by 6.64.

Hence $N = G$ by (i) and so $S \lhd G$. It follows that K is nilpotent, $K \leq F = \mathfrak{N}G$. A fortiori Φ is nilpotent. Since F/Φ is nilpotent and normal in G/Φ, we have $F \leq \mathfrak{N}(G \bmod \Phi) = K$. Combining gives $F = K$.

We show that F/Φ is Abelian by proving a more precise result.

<u>Theorem 9.2</u> Let M be a maximal subgroup of G, let $L = K_G(M) = \bigcap_{\xi \in G} M^\xi$ and let $N = \mathfrak{N}(G \bmod L)$. Then either $N = L$ or else N/L is a chief factor of G. In the latter case, N/L is an elementary Abelian p-group for some p and is the only minimal normal subgroup of G/L. Further, $N = C_G(N/L)$ and $M \cap N = L$, so that M/L is represented faithfully by automorphisms of N/L. Also $MN = G$ so that $|G : M| = |N : L| = p^n$ for some n.

<u>Corollary</u>. In a soluble group, every maximal subgroup is of index a power of a prime.

Proof: Suppose that $N > L$ and let P/L be any chief factor of G such that $P \leq N$. P/L cannot be semisimple since N/L is nilpotent. Hence P/L is an elementary p-group for some p, by 8.93. Let $Q = M \cap P$ so that $L \leq Q \lhd M$. Since L is the greatest normal subgroup of G contained in M, and $L < P \lhd G$, we have $P \not\leq M$ and so $PM = G$ by the maximality of M. Since P/L is Abelian, $Q \lhd P$ and so $Q \lhd PM = G$. But $L \leq Q < P$ and P/L is a chief factor of G. Hence $Q = L$.

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Since N/L is nilpotent and $P \lhd G$, we have $L_1 = L[P, N] < P$ and $L_1 \lhd G$, hence $L_1 = L$. So $N \leq C = C_G(P/L)$. Let $D = C \cap M$. Then $[D, P] \leq L \leq D$ and so $D \lhd PM = G$. Hence $D = L$ and so $N = P = C$. Thus we obtain $N = C_G(N/L)$, $M \cap N = L$, $MN = G$. If R/L is any minimal normal subgroup of G/L distinct from N/L, we should have $[R, N] \leq R \cap N = L$ and so $R \leq C_G(N/L) = N$, a contradiction. Thus the Fitting subgroup N/L of G/L is the only minimal normal subgroup of G/L and all is proved.

If $F = \mathfrak{N}G$, then $LF/L \cong F/L \cap F$ which is nilpotent and so $F \leq N$ and $F' \leq N' \leq L \leq M$. This is true for every maximal subgroup M

of G. Hence $F' \leq \Phi = \varphi(G)$ and the proof of 9·1 is now also complete. Indeed we see that the Sylow subgroups of F/Φ are all elementary Abelian. 9·1 and 9·2 are due to E. Galois 1811-32, G. Frattini 1852-1925 and W. Gaschütz.

. Theorem 9·3 Let H/K be a chief factor of G and suppose that $|H:K| = p^n$. Then (i) $\mathfrak{N}(G \bmod K) \leq C = C_G(H/K)$ and (ii) the automizer $A = A_G(H/K) \cong G/C$ of H/K in G has no normal p-subgroup $\neq 1$.

Proof: (i) The proof that $N = \mathfrak{N}(G \bmod K) \leq C$ has already occurred in the course of proving 9·2.

(ii) Let B be any p-subgroup of A. Since B leaves invariant the unit element of H/K and $|H:K| = p^n$, B must also leave invariant further elements of H/K, by 5·1. In the isomorphism of A with G/C, let B correspond to D/C. If $B \triangleleft A$, then $D \triangleleft G$ and the set of all elements $\xi \in H$ such that $K\xi$ is invariant under B, or equivalently $[\xi, D] \leq K$, is also a normal subgroup H_1 of G. Since $K < H_1 \leq H$, it follows that $H_1 = H$ and so $[H, D] \leq K$, $D \leq C$, $B = 1$.

(B). Let f be any representation of a group H by automorphisms of another group K, so that $f(\eta) \in \operatorname{Aut} K$ for $\eta \in H$ and $f(\eta_1\eta_2) = f(\eta_1)f(\eta_2)$. It is convenient to write the ordered pairs (η, ξ) with $\eta \in H$, $\xi \in K$ as formal products $\eta\xi$, the force of 'formal' being that $\eta_1\xi_1 = \eta_2\xi_2$ if and only if $\eta_1 = \eta_2$ and $\xi_1 = \xi_2$. If we define in the set G of all such formal products a multiplication $(\eta_1\xi_1)(\eta_2\xi_2) = \eta_3\xi_3$ by the rule that

$$\eta_3 = \eta_1\eta_2 \quad \text{and} \quad \xi_3 = \xi_1^{f(\eta_2)}\xi_2 , \tag{1}$$

then G becomes a group. If we identify the formal product $\eta 1$ with the element η of H and similarly 1ξ with the element ξ of K, then H and K become subgroups of G such that

$$HK = G, \quad H \cap K = 1 \quad \text{and} \quad K \triangleleft G. \tag{2}$$

Moreover in G, the automorphism $t_K(\eta)$ of K induced by transforming K by the element η of H is precisely $f(\eta)$.

G is called the split (or complemented) extension of K by H

determined by the representation f, and we shall denote it by

$$G = \langle H, K ; f \rangle. \qquad (3)$$

Given $\alpha \in \mathrm{Aut}\, K$ and $\beta \in \mathrm{Aut}\, H$, we obtain from f a new representation f^* of H by automorphisms of K, defined by

$$\xi^{f^*(\eta)} = \xi^{\alpha^{-1} f(\eta^\beta)\alpha} \qquad (\eta \in H,\ \xi \in K). \qquad (4)$$

And we have

$$G^* = \langle H, K ; f^* \rangle \cong \langle H, K ; f \rangle, \qquad (5)$$

the mapping $\eta\xi \to \eta^\beta \xi^{\alpha^{-1}}$ of G^* onto G being an isomorphism.

However, it is sometimes possible for there to be an isomorphism of the form (5) even when f and f^* are not related as in (4). The reason for this is twofold. First, it may happen that G has a normal subgroup $K_1 \neq K$ such that $K_1 \cong K$ and $G/K_1 \cong G/K$, and such that K_1 is complemented in G by a subgroup H_1. Then $H_1 \cong H$. If $\xi \to \xi_1$ and $\eta \to \eta_1$ are isomorphisms of K onto K_1 and H onto H_1 and if we define f^* by the equation $\xi_1^{\eta_1} = (\xi^{f^*(\eta)})_1$, then f^* is a representation of H by automorphisms of K and $\langle H, K ; f^* \rangle \cong \langle H, K ; f \rangle$. However, this possibility can be excluded if K is an S_ϖ-subgroup of G i.e. if $|H|$ and $|K|$ are coprime, for then $K_1 = K$ by 5.8.

If there is no such subgroup $K_1 \neq K$, we have only to consider the subgroups H_1 which are complementary to K in G. Every such H_1 contains exactly one element $\eta k(\eta)$ in the coset $\eta K = K\eta$ of K in G. Here η is any element of H and $k(\eta) \in K$. Since H_1 is a subgroup, we have $\eta_1 \eta_2 k(\eta_1\eta_2) = \eta_1 k(\eta_1) \eta_2 k(\eta_2)$ and so

$$k(\eta_1\eta_2) = (k(\eta_1))^{\eta_2} k(\eta_2) \qquad (\eta_1, \eta_2 \in H). \qquad (6)$$

Any mapping k of H into K satisfying (6) is called a <u>cocycle</u> with respect to f. The subgroups H_1 complementary to K in G are therefore in one-to-one correspondence with these cocycles. Among these subgroups there occur the conjugates of H in G. Every such conjugate has the form $H_1 = H^\kappa$ with $\kappa \in K$ and for this case the cocycle is defined by

$$k(\eta) = [\eta, \kappa]; \qquad (7)$$

and the representation f^* of H which we obtain by replacing each $\eta \in H$ by $\eta^\kappa \in H$, is one of those given by (4), viz. with $\alpha = t_K(\kappa)$ and $\beta = 1$.

We define

$$c_1 = c_1(H, K; f) \qquad (8)$$

to be the number of classes of conjugates in G into which the subgroups H_1 complementary to K split up. If G has no subgroup $K_1 \neq K$ of the kind considered above and if in addition $c_1 = 1$, then we can affirm that $G = \langle H, K; f\rangle \cong \langle H, K; f^*\rangle$ if and only if f^* is related to f by (4), for some choice of α and β. If this simple case occurs for all choices of the representation f, then it is an easy matter to determine in principle how many non-isomorphic split extensions of K by H exist; although in practice this involves a good understanding of the structure of H and Aut K.

(C). We now prove an important result due to H. Zassenhaus, though attributed by him to Schur. This is

Theorem 9.4 Let K be a normal S_ϖ-subgroup of G. (i) Then G contains an $S_{\varpi'}$-subgroup H, so that $K \cap H = 1$, $HK = G$, $H \cong G/K$. If either H or K is soluble, then every $S_{\varpi'}$-subgroup of G is conjugate to H in G.

Proof: (i) Suppose that G is a group of least order for which the result is not true, and let S be a Sylow p-subgroup of K and $N = N_G(S)$. Then $KN = G$ since $K \triangleleft G$, and $K \cap N$ is a normal S_ϖ-subgroup of N. If $N < G$, there is a subgroup H complementary to $K \cap N$ in N, by choice of G. Since $KN = G$, H is an $S_{\varpi'}$-subgroup of G, contrary to hypothesis. Hence $N = G$. This is true for all p, so that K is nilpotent. Let $Z = \mathfrak{z}K$. Then Z char K and so $Z \triangleleft G$. K/Z is a normal S_ϖ-subgroup of G/Z and, since $Z \neq 1$, there is a subgroup $G_1 \geq Z$ such that $G = G_1K$, $Z = G_1 \cap K$. Then Z is a normal S_ϖ-subgroup of G_1 and, if $G_1 < G$, there is a complement H to Z in G_1. Since $G = G_1K$, H is an $S_{\varpi'}$-subgroup of G contrary to hypothesis.

Hence $G_1 = G$ and $K = Z$ is Abelian.

Now T be any transversal to K in G. We label the elements $t_\alpha \in$ by the coset $\alpha \in G/K$ to which they belong. Then $t_\alpha t_\beta \in \alpha\beta$ and s $t_\alpha t_\beta = t_{\alpha\beta}\, k(\alpha,\beta)$ where $k(\alpha,\beta) \in K$ for all $\alpha, \beta \in G/K$. If $\gamma \in G/K$ also, we have $(t_\alpha t_\beta) t_\gamma = t_\alpha (t_\beta t_\gamma)$. But K is Abelian and so the automorphism $\xi \to t_\alpha^{-1} \xi t_\alpha$ $(\xi \in K)$ depends only on α. Writing ξ^α for $t_\alpha^{-1} \xi t_\alpha$, we find that the function $k(\alpha,\beta)$ satisfies

$$k(\alpha\beta,\gamma)\, k(\alpha,\beta)^{\gamma} = k(\alpha,\beta\gamma)\, k(\beta,\gamma) \tag{9}$$

for all α, β, γ in G/K. Defining $k_1(\beta) = \prod_{\alpha \in G/K} k(\alpha,\beta)$ and forming the products of the two sides of (9) as α runs through G/K, with β and γ kept fixed, we obtain

$$k_1(\gamma)\, k_1(\beta)^{\gamma} = k_1(\beta\gamma)\, k(\beta,\gamma)^{n} \tag{10}$$

where $n = |G:K|$. If $m = |K|$, then $(m,n) = 1$ by hypothesis. Hence there is an integer l such that $ln \equiv 1 \mod m$. We then have $k(\beta,\gamma) = k(\beta,\gamma)^{nl} = k_2(\beta\gamma)\, k_2(\gamma)^{-1} (k_2(\beta)^{\gamma})^{-1}$ where $k_2(\beta) = k_1(\beta)^{-l}$. Hence if $s_\beta = t_\beta\, k_2(\beta)$ for $\beta \in G/K$, we obtain $s_\beta s_\gamma = t_{\beta\gamma}\, k(\beta,\gamma)\, k_2(\beta)^{\gamma}$ $= t_{\beta\gamma}\, k_2(\beta\gamma) = s_{\beta\gamma}$. Hence the set S of all s_β $(\beta \in G/K)$ is a subgroup complementary to K in G. This final contradiction proves (i).

(ii) Let H and H_1 be two subgroups complementary to K in G. We assume that H and H_1 are not conjugate in G and choose G as sma as possible subject to this hypothesis. Let $L \lhd G$ and $1 < L < K$. Then either $H \cong G/K \cong (G/L)/(K/L)$ is soluble or else K/L is soluble. Hence, by our minimal choice of G, HL and H_1L are conjugate in G and we may assume $HL = H_1L = G_1$. Since either H or L is soluble, H and H_1 are conjugate in G_1, again by choice of G. This is a contradiction. Hence K is a minimal normal subgroup of G.

Let $M = O_{\varpi'}G$ be the maximal normal ϖ'-subgroup of G. Then H an H_1 both contain M. If $M \neq 1$, it follows that H/M and H_1/M are conjugate in G/M by choice of G; and hence H and H_1 are conjugate in G, contrary to hypothesis. Hence $M = 1$.

Suppose first that H is soluble and let P/K be a minimal normal subgroup of $G/K \cong H$. Then P/K is an elementary p-group and $P = KQ$ where Q $= P \cap H$ is a Sylow p-subgroup of P. $Q_1 = P \cap H_1$ is also a Sylow p-subgroup of P. These are conjugate in P and so we may assume that $Q = Q_1$. Then H and H_1 are $S_{\varpi'}$-subgroups of $N = N_G(Q)$. If $N < G$, H and H_1 are conjugate in N, since N has the normal S_ϖ-subgroup $N \cap K$. Hence $N = G$ and so P is the direct product of K and Q. But then Q char P and so $Q \lhd G$. This contradicts $M = 1$ and we conclude that H must be insoluble.

It now follows from the hypothesis that K is soluble. As a minimal normal subgroup of G, K is therefore an elementary Abelian q-group for some prime q. Let $\eta \in H$ and let $\eta k(\eta)$ be the element of H_1 in ηK so that k is a cocycle satisfying equation (6); or

$$k(\eta\zeta) = k(\eta)^\zeta k(\zeta) \qquad (\eta, \zeta \in H). \qquad (11)$$

We then have $\alpha = \prod_{\eta \in H} k(\eta) = \prod_{\eta \in H} k(\eta\zeta) = \alpha^\zeta k(\zeta)^n$, where $n = |H|$. As in (i) we choose l so that $ln \equiv 1 \mod m = |K|$ and obtain $\alpha^l = \beta = \beta^\zeta k(\zeta)$ for all $\zeta \in H$, whence $\zeta^\beta = \zeta k(\zeta)$ and so $H_1 = H^\beta$. Hence $\beta \in K$. This final contradiction proves (ii).

(D) We now consider some corollaries of 9.4.

Corollary 9.41. Let Γ be a group of automorphisms of G of order prime to $|G|$. If either G or Γ is soluble, then for each prime p, Γ leaves invariant some Sylow p-subgroup of G.

We apply 9.4 (ii) to the split extension ΓG. If S is any Sylow p-subgroup of G and $N = N_{\Gamma G}(S)$, then $NG = \Gamma G$ and so by 9.4 (i), N contains a subgroup Γ_1 complementary to $N \cap G$. By 9.4 (ii), Γ_1 is conjugate to Γ in ΓG. Hence Γ also leaves invariant some Sylow p-subgroup of G.

A group G is called ϖ-<u>seperable</u> if the composition factors of G are all either ϖ-groups or ϖ'-groups. G is called ϖ-soluble if the composition factors of G are all either ~~ϖ~~ p-groups for some $p \in \varpi$ or else ϖ'-groups. Note that ϖ-seperable $\Leftrightarrow$ ϖ'-separable, ϖ-soluble $\Rightarrow$ ϖ-seperable and p-soluble $\Leftrightarrow$ p-seperable. Also, in the above definitions the composition factors could be replaced by the chief factors. ~~[illegible]~~ G is ϖ-seperable if & only if it has a series whose factors are all either ϖ-groups or ϖ'-groups. G is ϖ-soluble if and only if it has a series whose factors are all either p-groups with some $p \in \varpi$ or else ϖ'-groups. Subgroups, quotient groups and sections of ϖ-separable groups are ϖ-seperable. Similarly for ϖ-soluble.

<u>Lemma 9.5</u> Let $p \in \varpi$, $q \in \varpi'$ and let G be ϖ-seperable. Then G has S_ϖ-subgroups, $S_{\varpi q}$-subgroups and S_{pq}-subgroups.

Since ϖ-separable $\Rightarrow$ ϖ'-seperable, it follows that G also has $S_{\varpi'}$-subgroups and $S_{p\varpi'}$-subgroups.

Proof by induction on $|G|$. We may assume $G \neq 1$. Let M be a minimal normal subgroup of G, and let H/M be an S_ψ-subgroup of G/M with $\psi = \varpi$, ϖq or pq. H exists by the induction hypothesis, and if $H < G$ the result follows at once for G. Hence we may assume G/M is a ψ-group. ~~[illegible]~~

First let M be a ϖ-group. If $\psi = \varpi$ or ϖq, the result follows at once. If S is a Sylow p-subgroup of M and $G_1 = N_G(S) < G$, then $MG_1 = G$ and an S_{pq}-subgroup of M is an S_{pq}-subgroup of G. Hence we may assume $G_1 = G$, $S \lhd G$, $M = S$ and again the result is immediate.

If M is not a ϖ-group, it must be a ϖ'-group. When $\psi = \varpi$, M is then an $S_{\varpi'}$-subgroup of G and the result follows from 9.4. When $\psi = \varpi$ let T be a Sylow q-subgroup of M and $G_2 = N_G(T)$. If $G_2 < G$, an $S_{\varpi q}$-subgroup of G_2 exists by induction and is an $S_{\varpi q}$-subgroup of G since $MG_2 = G$. If $G_2 = G$, then $T \lhd G$, $T = M$ and the result is immediate. The case $\psi = pq$ is symmetrical as between ϖ and ϖ' and has already been dealt with.

<u>Lemma 9·52</u> Let G be either ϖ-soluble or ϖ'-soluble, let H be any S_ϖ-subgroup of G and let L be any ϖ-subgroup of G. Then $L^\xi \leq H$ for some $\xi \in G$. In particular, any two S_ϖ-subgroups of G are conjugate in G.

Proof by induction on $|G|$. We can assume $G \neq 1$. Let M be a minimal normal subgroup of G. Then MH/M is an S_ϖ-subgroup of G/M and ML/M is a ϖ-subgroup. By induction, $L^\eta \leq MH$ for some $\eta \in G$. The theorem follows by a second induction if $MH < G$, since H is an S_ϖ-subgroup of MH. So we may assume $MH = G$. Then G/M is a ϖ-group. If M is a ϖ-group, then $H = G$ and there is nothing left to prove. If M is not a ϖ-group, it is a normal $S_{\varpi'}$-subgroup of G, since G is ϖ-seperable. Further, $H \cap M = 1$, $H \cong G/M$ and by hypothesis at least one of the groups M and H is soluble. ~~By~~ Also $L \cap M = 1$ and if $G_1 = LM$, $L_1 = H \cap G_1$, then L and L_1 are S_ϖ-subgroups of G_1. Either L or M is soluble. Hence $L_1 = L^\xi$ for some $\xi \in M$ by 9.4 (ii) and the lemma follows.

For soluble groups, we may now state a result analogous to Sylow's Theorem but with the prime p replaced by an arbitrary set of primes ϖ. This is

<u>Theorem 9·5</u> Let G be soluble and let ϖ be any set of primes. Then (i) G contains S_ϖ-subgroups.

(ii) Any two S_ϖ-subgroups of G are conjugate in G

(iii) Every ϖ-subgroup of G is contained in some S_ϖ-subgroup of G

By 9.52, if G is either ϖ-soluble or ϖ'-soluble, the S_ϖ-subgroups of G may be described alternatively as the maximal ϖ-subgroups of G. It also follows that <u>any S_ϖ-subgroup H of G is pronormal in G</u> and <u>$N = N_G(H)$ is abnormal</u> in G.

(E) Lemma 9.6 Let H and K be subgroups of G, ~~...~~ and $L = H \cap K$. Then (i) $|K:L| \leq |G:H|$ and so $|G:L| \leq |G:H|.|G:K|$

(ii) If H and K are conjugate in G, and distinct, then $|K:L| < |G:H|$ and $HK < G$

(iii) If $|G:H| = m$ and $|G:K| = n$ are coprime, then $|G:L| = mn$ and $HK = G$.

Proof: (i) $|K:L| = |HK:H|$ by 5.5.

(ii) If $\eta \in H$, $\zeta \in K$, then $H^{\eta\zeta} = H^{\zeta} \neq K^{\zeta} = K$. Hence if H and K are conjugate in G, and distinct, then $HK < G$ and so $|K:L| < |G:H|$

(iii) Since $L \leq H \leq G$, $|G:L|$ is a multiple of m. Similarly it is a multiple of n. Since $(m,n) = 1$, $|G:L|$ is a multiple of mn. But $|G:L| \leq mn$ by (i). Hence $|G:L| = mn$ and so $|K:L| = m$ and $HK = G$.

Theorem 9.7 Suppose that G has ~~...~~ three soluble ~~...~~ subgroups H_1, H_2, H_3 whose indices m_1, m_2, m_3 in G are coprime in pairs. Then G is soluble.

Proof: If $H_1 = 1$, then $H_2 = H_3 = G$ and there is nothing to prove. So we can assume $H_1 \neq 1$. Let M be a minimal normal subgroup of H_1. Then $|M| = p^m$ for some prime p. Since $(m_2, m_3) = 1$, we may assume that p does not divide m_2. By 9.6 (iii), if $L_{12} = H_1 \cap H_2$, then $|H_1 : L_{12}| = m_2$. Hence L_{12} contains a Sylow p-subgroup of H_1 and so $M \leq L_{12}$. Also $G = H_1 H_2$ and so every element of G has the form $\xi\eta$ with $\xi \in H_1$, $\eta \in H_2$. This shows that every conjugate $M^{\xi\eta} = M^{\eta}$ of M in G is contained in H_2. Hence $K = \{M^{\eta}; \eta \in H_2\}$ is normal in G and soluble. In G/K, the subgroups $KH_i/K \cong H_i/K \cap H_i$ are soluble and their indices m_i' are coprime in pairs since $m_i' = |G:KH_i|$ divides $m_i = |G:H_i|$. By induction on $|G|$, we may assume that G/K is soluble. Hence G is soluble.

(F). The proof of the following theorem of Burnside depends on the theory of group characters and will be postponed to §   .

<u>Theorem 9.8</u> If the group G has a class of $p^n > 1$ conjugate elements, then G is not simple.

<u>Corollary</u>. If $|G|$ is divisible by only two primes p and q, then G is soluble.

Proof by induction on $|G|$. We may assume that $G \neq 1$ and that $\mathfrak{z}G = 1$. Then G is not a p-group. Let Q be a Sylow q-subgroup of G. Then $Q \neq 1$ and so $\mathfrak{z}Q \neq 1$. Let $\xi \in \mathfrak{z}Q$, $\xi \neq 1$. Since $\mathfrak{z}G = 1$, we have $|G : C_G(\xi)| = p^n > 1$. Hence G is not simple by 9.4. So there is a subgroup $K \lhd G$ with $1 < K < G$. By induction both K and G/K are soluble. Hence G is soluble.

<u>Theorem 9.9</u> A group G is soluble if and only if it has an $S_{p'}$-subgroup for every prime p.

Proof: the condition is necessary by 9.5. Let $|G| = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1 < p_2 < \cdots < p_k$ are primes. If $k \leq 2$, then G is soluble by 9.8. Suppose that G has an $S_{p_i'}$-subgroup H_i for each $i = 1, 2, \ldots, k$. If $k =$ each H_i is soluble by 9.8 and so G is soluble by 9.7. Let $k > 3$. By 9.6(iii), if $i \neq j$, $H_i \cap H_j$ is an $S_{p_j'}$-subgroup of H_i. So we may assume each H_i is soluble by induction on k. It now follows that G is soluble by 9.7 again.

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