§11. Subnormal Subgroups (Wielandt)

(A). If H is any subgroup of G, we denote by $H^{\cdot\cdot G}$ the <u>subnormal closure</u> of H in G i.e. the intersection of all the subnormal subgroups of G which contain H. By 6·3 (iii), we have

$$H^{\cdot\cdot G} \text{ sbn } G .$$

The normal closure of H in $H^{\cdot\cdot G}$ is $H^{\cdot\cdot G}$ itself. Define $H_0 = G$ and $H_{n+1} = \{H^{H_n}\}$ inductively for $n \geqslant 0$. Then for some $r \geqslant 0$, we have

$$G = H_0 > H_1 > \cdots > H_r = H_{r+1} .$$

Since $H_{n+1} \lhd H_n$ for all n, we have H_r sbn G and so $H^{\cdot\cdot G} \leqslant H_r$. By 6·3 $H^{\cdot\cdot G}$ sbn H_r. But $H_r = \{H^{H_r}\} = \{(H^{\cdot\cdot G})^{H_r}\}$. Hence $H_r = H^{\cdot\cdot G}$.

H sbn G if and only if $H = H^{\cdot\cdot G}$. In this case we call

$$G = H_0 \rhd H_1 \rhd \cdots \rhd H_r = H \tag{1}$$

the <u>canonical series from G to H</u> and write $r = m(G,H)$. If we have any series $G = G_0 \rhd G_1 \rhd \cdots \rhd G_s = H$ from G to H, then we obtain $G_i \geqslant H_i$ for all i by induction on i, and so $s \geqslant m(G,H)$. $m(G,H)$ is the minimal length of a series from G to H.

<u>Theorem 11·1</u> Let H and K be subnormal in G. Then $J = \{H, K\}$ is subnormal in G.

Proof: Suppose first that $H \lhd J$. Then K normalizes not only H but also, by induction on i, every term H_i of the canonical series (1). Hence $K_i = H_i K$ is a subgroup of G. Since K sbn G, we have K sbn K_i by 6·2. But $H_{i+1} \lhd H_i$ and K normalizes H_{i+1}. Hence $H_{i+1} \lhd K_i$. Now the product of a normal subgroup with a subnormal subgroup is a subnormal subgroup, since epimorphisms preserve the normality relation. Hence $K_{i+1} = H_{i+1}K$ sbn K_i. By 6·3 (ii), it follows that $J = K_r = HK$ sbn $G = K_0$ as required.

We prove the general result by induction on $m(G,H) = r$. If $r = 1$, then $H \lhd G$ and so $J = HK$ sbn G. Suppose $r > 1$, and let $\overline{H} = \{H^J\}$. For any $\xi \in J$, we have H^ξ sbn $H_1^\xi = H_1$ and $m(H_1, H^\xi) = m(H_1, H) = r-1$

By the induction on r, all the groups $M = \{H^{\xi_1}, H^{\xi_2}, \dots, H^{\xi_n}\}$ with $\xi_1, \dots, \xi_n$ in J are subnormal in H, and hence in G. This follows by induction on n. But for a suitable choice of the ξ's, $M = \bar{H}$. Hence $\bar{H}$ sbn G. But $J = \bar{H}K$ and $\bar{H} \lhd J$, K sbn G. So by the special case already considered we obtain J sbn G as required.

In view of 6·3 (iii), we may now state as a corollary that <u>the subnormal subgroups of any group G form a sublattice of the lattice of all subgroups of G.</u>

(B). By 7·61, every subnormal nilpotent subgroup of G is contained in the Fitting subgroup $\mathfrak{N}G$ of G. Conversely, if H is any subgroup of $\mathfrak{N}G$, then H sbn $\mathfrak{N}G$ by 6·8 (iii). Since $\mathfrak{N}G$ char G, it follows that H sbn G. Thus <u>the subnormal nilpotent subgroups of G are precisely the subgroups of the Fitting subgroup of G.</u>

<u>Theorem 11·2</u> (i) Let H sbn G and let P be a Sylow p-subgroup of G. Then $H \cap P$ is a Sylow p-subgroup of H.

(ii) Let H be a soluble subgroup of G and suppose that, for all primes p and every Sylow p-subgroup P of G, $H \cap P$ is always a Sylow p-subgroup of H. Then H sbn G.

Proof: (i) is a corollary of 5·6.

(ii) Assuming $H \neq 1$, let M be a minimal normal subgroup of H. Since H is soluble, $|M| = p^n$ for some prime p and every Sylow p-subgroup of H contains M. Hence every Sylow p-subgroup of G contains M and $M \leq \bar{M}$, the maximal normal p-subgroup of G. Let $L = H\bar{M}$ and let Q_1 be a Sylow q-subgroup of L where $q \neq p$. Then $Q_1 \leq Q$, where Q is some Sylow q-subgroup of G and we have $H \cap Q \leq L \cap Q = Q_1$. By hypothesis, $H \cap Q$ is a Sylow q-subgroup of H. But $\bar{M} \lhd L$ and $Q \cap \bar{M} = 1$, since $q \neq p$. Since $L = H\bar{M}$, it follows that $H \cap Q$ is a Sylow q-subgroup of L. Hence $H \cap Q = Q_1 \leq H$. Thus H contains all p'-elements of L. These form a normal subgroup K of L and L/K is a p-group. Since $K \leq H \leq L$ it follows that

H sbn L, (since L/K is nilpotent by 6·8.)

The Sylow subgroups of $G/\overline{M}$ have the form $\overline{M}S/\overline{M}$ where S is a Sylow subgroup of G, by 5·6. Since $H \cap S$ is a Sylow subgroup of H by hypothesis, it follows that $(L \cap \overline{M}S)/\overline{M}$ is a Sylow subgroup of $L/\overline{M}$. This is true for every Sylow subgroup $\overline{M}S/\overline{M}$ of $G/\overline{M}$. By induction on $|G|$, we may therefore assume that $L/\overline{M}$ sbn $G/\overline{M}$, since $L/\overline{M} \cong H/\overline{M}$ is soluble. Thus L sbn G. Since H sbn L, we now have H sbn G by 6·3(ii).

Problem: let p be a fixed prime. What can be said of those (soluble) subgroups H of G which intersect every Sylow p-subgroup of G in a Sylow p-subgroup of H?

(C) Theorem 11·3 (i) Let H sbn G, let P be a Sylow p-subgroup of H and let L be any subgroup of G. Then $\{P, L\}$ contains a Sylow p-subgroup of $\{H, L\}$.

(ii) Let $H_1, H_2, \dots, H_n$ be subnormal subgroups of G and let P_i be a Sylow p-subgroup of H_i. Then $Q = \{P_1, P_2, \dots, P_n\}$ contains a Sylow p-subgroup of $K = \{H_1, H_2, \dots, H_n\}$.

(iii) The mapping

$$H \to H \cap P \qquad (H \text{ sbn } G),$$

where P is a fixed Sylow subgroup of G is a lattice homomorphism; in particular, if H_1 and H_2 are subnormal in G, then

$$\{H_1, H_2\} \cap P = \{H_1 \cap P, H_2 \cap P\}.$$

Proof: (i) is immediate if $H = G$ since then P is a Sylow p-subgroup of G. If $H < G$, then $\overline{H} = \{H^\xi ; \xi \in L\} < G$ and H^ξ sbn $\overline{H}$ for all $\xi \in L$ by 6.2. Also P^ξ is a Sylow p-subgroup of H^ξ. By induction on $|G|$, we may assume (ii) holds in $\overline{H}$ and so $\overline{P} = \{P^\xi ; \xi \in L\}$ contains a Sylow p-subgroup of $\overline{H}$. But $\{H, L\} = \overline{H}L$; $\{P, L\} = \overline{P}L$; and by 3

$$|\overline{H}L : \overline{P}L| = |\overline{H} : \overline{P}| . \frac{|L : \overline{H} \cap L|}{|L : \overline{P} \cap L|}$$

divides $|\overline{H} : \overline{P}|$ and is therefore prime to p. Thus (i) is proved.

To prove (ii), first suppose $n = 2$. Then we have

$|K:Q| = |K:\{P_1, H_2\}| \cdot |\{P_1, H_2\}:Q|$ and both these factors are prime to p by (i) since H_1 and H_2 are subnormal in G. Now let $n > 2$. By induction on n, we may assume that $|K^*:Q^*|$ is prime to p, where $K^* = \{H_1, \ldots, H_{n-1}\}$ and $Q^* = \{P_1, P_2, \ldots, P_{n-1}\}$. But K^* sbn G by 11·1. Hence we have $|K:Q| = |\{K^*, H_n\}:\{Q^*, P_n\}|$ is prime to p by the case $n=2$, since Q^* contains a Sylow p-subgroup of K^*. Thus (ii) is proved.

(iii) $(H_1 \cap P) \cap (H_2 \cap P) = (H_1 \cap H_2) \cap P$ so the mapping $H \to H \cap P$ is homomorphic with respect to intersections. If H_1 and H_2 are subnormal in G, then $H_1 \cap P$ and $H_2 \cap P$ are respectively Sylow p-subgroups of H_1 and H_2 and so $Q = \{H_1 \cap P, H_2 \cap P\}$ contains a Sylow p-subgroup of $K = \{H_1, H_2\}$ by (ii). But Q is a p-group. Hence Q is a Sylow p-subgroup of K. But K sbn G by 11·1 and so $K \cap P$ is also a Sylow p-subgroup of K. Since $Q \leq K \cap P$, it follows that $Q = K \cap P$ and so the mapping $H \to H \cap P$ (H sbn G) is also homomorphic with respect to joins.

(D). Let $M = \mathcal{S}G$ be the semisimple radical of G and let H sbn G, so that
$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G. \qquad (1)$$
Then $M_i = \mathcal{S}H_i = H_i \cap M$ by 8.91 (ii) and M_i sbn G. Hence M_i is a direct factor of M by 8.91(i) and we may suppose that M_i is the direct product of M_{i-1} and N_i for each $i = 1, 2, \ldots, r$. Since H normalizes both M_i and M_{i-1}, it must also normalize N_i. But $[H, N_i] \leq [H_{i-1}, H_i] \leq H_{i-1}$ and so $[H, N_i] \leq N_i \cap H_{i-1} = 1$. If $N = N_1 N_2 \cdots N_r$, we then have $[H, N] = 1$ and $M = \mathcal{S}G$ is the direct product of $M_0 = \mathcal{S}H$ and N. Since M_0 is semisimple, it is clear that $N = M \cap C_G(H)$.

Consider next the case in which $M = O_p G$ is the intersection of the Sylow p-subgroups of G. Then $M_i = M \cap H_i = O_p H_i$ by 8.92. Let $K = H^{O_p}$ be the subgroup generated by the p'-elements of H. Now $M = M_r$ and $[M_i, H] \leq M_{i-1}$ for each i, as in the semisimple case. Hence $[M, K] \leq M_r$. Suppose that for some $i > 0$, we have $[M, K] \leq M_i$ but $[M, K] \not\leq M_{i-1}$. Then we can choose $\alpha \in M$ so that $[\alpha, K] \not\leq M_{i-1}$. If $\xi, \eta \in K$, we have $[\xi\eta, \alpha] = [\xi, \alpha]^{\eta} [\eta, \alpha]$ by 7.1 (i). Now $[\xi, \alpha] \in M_i$ by hypothesis, and so $[\xi, \alpha, \eta] \in M_{i-1}$. Hence the mapping $\xi \to [\xi, \alpha] M_{i-1}$ $(\xi \in K)$ is a homomorphism of K into the p-group M_i / M_{i-1}. But $K = H^{O_p}$ and so K has no quotient p-group other than K/K. Therefore the kernel of this homomorphism must be K and so $[K, \alpha] \leq M_{i-1}$, which is a contradiction. It follows that $[M, K] \leq M_0$.

Thus we have proved

<u>Theorem 11·3</u> Let H be a subnormal subgroup of G. Then
(i) $\mathcal{S}G$ is the direct product of $\mathcal{S}H$ with $\mathcal{S}G \cap C_G(H)$.
(ii) For every prime p, $[O_p G, H^{O_p}] \leq O_p H$.

(E) <u>Lemma 11.41</u> Let H, K, L, M be subgroups of a group G such that $[K, H] \leq L$ and $C_K(H) \leq M$. Then $|K| \leq |L|^{|H|} \cdot |M|$

Given $\alpha \in K$ consider the mapping $k(\alpha) : \xi \to [\xi, \alpha]$ $(\xi \in H)$. This is a mapping of H into L. If $\beta \in K$, we have $k(\alpha) = k(\beta)$ if and only if $\xi^\alpha = \xi^\beta$ for all $\xi \in H$, i.e. if $\alpha\beta^{-1} \in C_K(H)$. The number of distinct mappings $k(\alpha)$, $\alpha \in K$, is therefore $|K : C_K(H)|$. This cannot exceed $|L|^{|H|}$. Since $|C_K(H)| \leq |M|$, we obtain the result stated.

<u>Theorem 11.42</u> Let H be a subnormal subgroup of G and let $C = C_G(H^{\mathfrak{N}})$, where $H^{\mathfrak{N}} = \mathfrak{G}_\infty H = \bigcap_p H^{\mathcal{O}_p}$ is the lower central limit of H. Then $|G|$ is bounded by a number depending only on $|H|$ and $|C|$.

Proof: Let $h = |H|$ and $c = |C|$. We can choose the classes $\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_n$ where each $\mathfrak{S}_i$ is either $\mathfrak{S}$ or $\mathcal{O}_p$ for some prime p so that we have

$$1 = L_0 < L_1 < \cdots < L_n = H$$

and

$$L_i = \mathfrak{S}_i (H \bmod L_{i-1}) \qquad (i = 1, 2, \dots, n).$$

Here n is at most equal to the length of a chief series of H and so $n < h$, if we ignore the trivial case $H = 1$. If we define

$$1 = K_0 < K_1 < \dots < K_n = K$$

by the equations $K_i = \mathfrak{S}_i (G \bmod K_{i-1})$, then by 8.92 we have $H_i = H \cap K_i$ for each $i = 1, 2, \dots, n$. If $N = H^{\mathfrak{N}}$, we have $N \leq H^{\mathcal{O}_p} \leq H$ and so $[K_i, N] \leq K_{i-1} L_i$ for each i, by 11.3. Let $k_i = |K_i|$. Since N and L_i are subgroups of H, we obtain $k_i \leq (k_{i-1} h)^h c$, by 11.41; and so by induction on i, $k = k_n \leq f(h, c)$ owing to $n < h$. Hence $|G : C_G(K)| \leq k!$, since $K \lhd G$. But $N \leq H \leq K$ and so $|C_G(K)| \leq c$. Hence $|G| \leq c \cdot f(h,c)!$ and 11.42 is proved.

Now suppose that $\mathfrak{S}_i = \mathcal{O}_p$ so that K_i / K_{i-1} is a p-group; and let $M = K_i \cap C_G(H^{\mathcal{O}_p})$. Then $M \lhd HM$, since $K_i \lhd G$ and H normalizes $C_G(H^{\mathcal{O}_p})$. Hence if Q is a Sylow p-subgroup of HM containing a given Sylow p-subgroup P of H, then $R = Q \cap M$ is a Sylow p-subgroup of M and $R \lhd Q$. If $R \neq 1$, it follows that there is an element $\zeta \neq 1$ in $R \cap \mathfrak{z}Q$. Then ζ centralizes H^{O_p}, since $\zeta \in M$; and ζ also centralizes P since $\zeta \in \mathfrak{z}Q$ and $P \leq Q$. But $H = PH^{O_p}$, and so ζ centralizes H. We conclude that, if $C_G(H) = 1$, then M must be a p'-group and hence $M \leq K_{i-1}$. In this case, 11·3(ii) and 11·4 give the inequality $k_i \leq (k_{i-1}\, h)^h k_{i-1}$; and so by induction on i, we find $k \leq f(h)$, a bound depending only on h. Since $H \leq K$, we also have $C_G(K) = 1$ and hence $|G| \leq k!$ Thus we obtain

<u>Theorem 11·43</u> Let H be subnormal in G and let $C_G(H) = 1$. Then $|G|$ is bounded by a number depending only on $|H|$.

(F). Let $H = H_1$ be a group with $\mathfrak{z}H = 1$. Then the mapping

$$\xi \to t(\xi) \qquad (\xi \in H)$$

of H onto its group of inner automorphisms is isomorphic. If we identify ξ with $t(\xi)$, we obtain $H_1 \lhd H_2 = \mathrm{Aut}\, H_1$. Since $[H_1, \mathfrak{z}H_2] = 1$, we have $\mathfrak{z}H_2 = 1$. Hence H_2 can be embedded canonically in $H_3 = \mathrm{Aut}\, H_2$ and so on. In this way we form the <u>automorphism tower</u> of H

$$H_1 \lhd H_2 \lhd H_3 \lhd H_4 \cdots$$

where $H_{i+1} = \mathrm{Aut}\, H_i$ and each $\xi \in H_i$ is identified with the corresponding inner automorphism $t(\xi) \in H_{i+1}$. And we have $\mathfrak{z}H_i = 1$ for all i.

For $i < j$, let $C_{ij} = C_{H_j}(H_i)$ and $N_{ij} = N_{H_j}(H_i)$. Then $C_{ij} \lhd N_{ij}$, $H_{i+1} \leq N_{ij}$ and $H_{i+1} \cap C_{ij} = 1$. Since $H_{i+1} \lhd H_{i+2}$, it follows that the product $H_{i+1} C_{i,i+2}$ is direct, and so $C_{i,i+2} \leq C_{i+1,i+2} = 1$. Every $\alpha \in N_{i,i+2}$ induces in H_i an automorphism $\beta \in H_{i+1}$ and so $\alpha\beta^{-1} \in C_{i,i+2}$. Since $C_{i,i+2} = 1$, it follows that $\alpha = \beta \in H_{i+1}$ and $N_{i,i+2} = H_{i+1}$. Suppose we have proved that $N_{i,i+j} = H_{i+1}$ and $C_{i,i+j} = 1$ for all $j = 1, 2, \cdots, k-1$ and all i. Then $H_{i+1} = N_{i,i+k} \cap H_{i+k-1} \lhd N_{i,i+k} \leq H_{i+k}$, since $H_{i+k-1} \lhd H_{i+k}$. Hence $N_{i,i+k} \leq N_{i+1,i+k} = H_{i+2}$ by hypothesis. So $N_{i,i+k} = N_{i,i+2} = H_{i+1}$, and ~~thus~~ so $C_{i,i+k} = C_{i,i+1} = 1$. By induction on k, we have proved

<u>Theorem 11·5</u> If $H = H_0 \lhd H_1 \lhd H_2 \lhd \cdots$ is the automorphism tower of a group H with $\mathfrak{z}H = 1$, then $N_{H_n}(H) = H_1$ and $C_{H_n}(H) = 1$ for all n. For some n, we must have $H_n = H_{n+1} = \cdots$

The last remark follows from 11·43.