Note that $H \lhd G$ by 5·2(v). Since $G' \neq 1$, the automizer $A_0 = A_G(H)$ is of order p.

Proof: (i) By 14·31 (iii), there is only one choice for A_0 viz. $A_0 = \{\alpha^{p^{n-3}}\}$. Here $\gamma = \alpha^{p^{n-3}}$ is $\xi \to \xi^{1+p^{n-2}} = \xi\zeta$. So $zG = \{\xi^p\}$ is of index p^2 and $G' = \{\zeta\}$ is of order p and G is of class 2. Also $G = \{\xi, \eta_1\}$ where $\eta_1^{-1}\xi\eta_1 = \xi\zeta$. Then $\eta_1^p = \xi^{rp}$ for some r, since η_1^p commutes with η_1. Since $[\xi^{-r}, \eta_1] = \zeta^{-r}$ and $\zeta^p = 1$, we have $(\eta_1\xi^{-r})^p = 1$ because p is odd. Then $G = \{\xi, \eta\}$ with $\eta^{-1}\xi\eta = \xi^{1+p^{n-2}}$, $\eta^p = 1$.
$K = \{\eta, \zeta\}$ is normal in G and elementary; $G = \{K, \xi\}$ so G/K is cyclic. No element outside of K can have order p, so $K = \Omega_1 G$.

(ii) By 14·31 (i), the choice of A_1 is again unique and $G = \{\xi, \eta\}$ with $\eta^{-1}\xi\eta = \xi^{-1}$. Here $G' = zG = \{\xi^2\}$ is of order 2 and there are only two possibilities for η^2 viz. $\eta^2 = 1$ giving O; or $\eta^2 = \xi^2$ giving Q. Since $(\eta\xi^r)^2 = \eta^2 \cdot \eta^{-1}\xi^r\eta \cdot \xi^r = \eta^2$, and $\eta\xi^r$ transforms ξ into its inverse for all choices of r, these two cases are distinct. In O, $H = \{\xi\}$ as the only cyclic subgroup of order 4 is characteristic; $|\mathrm{Aut}\, O| = 8$ because each of the mappings $\xi \to \xi^{\pm 1}$, $\eta \to \eta\xi^r$ $(r = 0, 1, 2, 3)$ defines an automorphism of O. If η^* is $\xi \to \xi^{-1}$, $\eta \to \eta$ and ξ^* is $\xi \to \xi$, $\eta \to \eta\xi$, then $\mathrm{Aut}\, O = \{\xi^*, \eta^*\}$ where ξ^* is of order 4, η^* of order 2 and η^* transforms ξ^* into its inverse. Thus $\mathrm{Aut}\, O \cong O$.

(iii) In the case of Q, all three subgroups of index 2 are cyclic. Hence if ξ_1, η_1 are any two non-commuting elements of Q, we have $Q = \{\xi_1, \eta_1\}$ and $\xi \to \xi_1$, $\eta \to \eta_1$ is an automorphism. Hence $|\mathrm{Aut}\, Q| = (8-2)(8-4) = 24$. If α is $\xi \to \eta \to \xi\eta$ and β is $\xi \leftrightarrow \eta^{-1}$, then $\alpha^3 = \beta^2 = 1$ and $\beta^{-1}\alpha\beta = \alpha^{-1}$ so $S = \{\alpha, \beta\} \cong \Sigma_3$. S permutes the three subgroups of index 2 in Q faithfully. These subgroups are normal in Q and so $\mathrm{Aut}\, Q = ST$, where $S \cap T = 1$ and T is the group of inner automorphisms of Q.
$T \cong Q/Q'$ is elementary of order 4, so $|\mathrm{Aut}\, T| = 6$. But $C_S(T) = 1$. Hence $\mathrm{Aut}\, Q$ is isomorphic with the holomorph of T. The same applies to Σ_4. So $\mathrm{Aut}\, Q \cong \Sigma_4$.

(iv). Here we have three possible choices for A_0 by 14·32, and $G = \{\xi, \eta\}$ where $\eta^{-1}\xi\eta$ is either $\xi^{1+2^{n-2}}$, ξ^{-1} or $\xi^{-1+2^{n-2}}$. Let $\zeta = \xi^{2^{n-2}}$. In the first case, as in (i) for p odd, we have $G' = \{\zeta\} \leqslant \mathfrak{z}G = \{\xi^2\}$ and G is of class 2, and again the maximal subgroups of G are all Abelian. If $\{\eta, \xi^2\}$ is cyclic, we may suppose $\eta^2 = \xi^2$ and obtain $(\eta\xi^{-1})^2 = 1$. If $\{\eta, \xi^2\}$ is not cyclic, it is of type $(n-2, 1)$. In any case we may choose $\eta = 1$. Thus G is determined to within isomorphism; and in fact just two of the three subgroups of index 2 are cyclic. The third is a characteristic Abelian subgroup L of type $(n-2, 1)$. Hence $\Omega_1 L$ contains all the involutions of G and is an elementary group of order 4, characteristic in G.

(v) In the remaining two cases, η transforms ξ^2 into its inverse and $\mathfrak{z}G = \{\zeta\}$ is of order 2. $[\eta, \xi] = \xi^2$ or $\xi^2\zeta$ and successive commutations with η give ξ^{-4}, ξ^{8}, ξ^{-16}, … . Hence G is of class $n-1$ and $\gamma_{k+1}G = \{\xi^{2^k}\}$. For η^2 only two values are available, 1 and ζ. However, when $\eta^{-1}\xi\eta = \xi^{-1}\zeta$ we have $(\eta\xi)^2 = \eta^2\zeta$ and the two choices for η^2 lead to the same group P_{2^n}, of intermediate, type. Taking $\eta^2 = 1$, $\{\xi^2, \eta\}$ is then of type $O_{2^{n-1}}$ and $\{\xi^2, \eta\xi\}$ of type $Q_{2^{n-1}}$. When $\eta^{-1}\xi\eta = \xi^{-1}$, we have $(\eta\xi^r)^2 = \eta^2$ for all r. When $\eta^2 = 1$, we have the group O_{2^n} in which every element outside $\{\xi\}$ is of order 2. When $\eta^2 = \zeta$, we have Q_{2^n} in which every element outside $\{\xi\}$ is of order 4. The remaining statements of (v) are clear.

<u>Corollary 14.33</u> If the p-group G has no elementary subgroup of order p^2, then either G is cyclic, or else $p = 2$ and $G = Q_{2^n}$ is the generalized quaternion for some $n \geqslant 3$. (Including the case $Q = Q_8$, $n = 3$, the ordinary quaternion group.)

Proof: Suppose G is not cyclic. Then it contains a non-cyclic subgroup G_1 with a cyclic subgroup of index p. G_1 cannot be Abelian, for then $\Omega_1 G_1$ would be elementary of order p^2. By 14·3 (i), it follows that $p = 2$. By 14·3 (ii), (iv), (v), G_1 must be of type Q_{2^n} for some n. By induction on n, we may assume $|G : G_1| = 2$. We may assume that

$G_1 = \{\xi, \eta\}$ with $\{\xi\}$ of order 2^{n-1} and normal in $G = \{G_1, \tau\}$; for $\{\xi\}$ char G_1 if $n > 3$ and if $n = 3$, G_1 has three cyclic subgroups of order 4. In Aut $\{\xi\}$, the automorphism $\xi \to \xi^{-1}$ induced by η is not a square. Hence $\tau^2 \in \{\xi\}$, and $G/\{\xi\}$ is elementary of order 4. Since the automorphism $\xi \to \xi^{1+2^{n-2}}$ cannot be induced in G, 14·32 shows that G has an Abelian subgroup H of index 2 viz. $H = C_G(\xi)$. Then H must be cyclic and so G is of type $Q_{2^{n+1}}$

(D). <u>Theorem 14·~~3~~4</u> Let G be a group of order 2^n, $n > 3$. Then the following conditions are equivalent.

(i) $|G : G'| = 4$.

(ii) G is of class $n-1$.

(iii) G is one of the groups O_{2^n}, P_{2^n}, Q_{2^n}.

Proof: (ii) ⇒ (i) is clear. For a nilpotent group G with cyclic G/G' must itself be cyclic, by 9·1 (i) and 6·8 (v). [This should have been made explicit in §9 (A) somewhere]. Hence $|G : G'| \geq 4$. But $|G : G'| > 4$ implies $\gamma_{n-1} G = 1$, so that the class of G would be less than $n-1$.

(i) ⇒ (iii). By 14·21 and 14·3 (ii), O and Q are the only non-Abelian groups of order 8. ~~[illegible]~~ Let $Z \lhd G$, $|Z| = 2$, $Z \leq G'$. Such a subgroup Z exists by ~~[illegible]~~ 5·2 (i), and by induction we may assume that G/Z has a cyclic subgroup H/Z of index 2. Let $Z = \{\zeta\}$ and $H = \{\xi, \zeta\}$. If $H = \{\xi\}$, the result follows from 14·3 (iv) and (v). If H is not cyclic, it is Abelian of type $(n-2, 1)$ and if $\eta = \xi^{2^{n-3}}$, then $Y = \{\eta\}$ is a characteristic subgroup of H. So $Y \lhd G$. Also $Y \leq G'$ since G/G' is elementary. By induction, G/Y also has a cyclic subgroup L/Y of index 2. But H/Y is Abelian of type $(n-3, 1)$. Hence $H \neq L$ and $G = HL$ and $H \cap L = \mathfrak{z}G$ is of index 4. Hence G is of class 2 and $G' = \{[\alpha, \beta]\}$ where $G = \{\alpha, \beta\}$. Here $\gamma = [\alpha, \beta]$ is of order 2 since $\alpha^2 \in \mathfrak{z}G$. Thus $|G'| = 2$, contradicting $n > 3$.

(iii) ⇒ (ii) is contained in 14·3 (iv), (v).

(E) We consider now the non-Abelian groups of order p^3. If $p=2$, then 14·21 and 14·3 (ii) show that the octic and quaternion groups are the only possible types. For odd p there are also exactly two distinct types of group.

For let $|G| = p^3$, $G' \neq 1$, p odd. If G has an element ξ of order p^2, then $G = \{\xi, \eta\}$ with $\eta^p = 1$, $\eta^{-1}\xi\eta = \xi^{1+p}$ by 14·3 (i). On the other hand, if G is of exponent p, then $G = \{\xi, \eta\}$ with $\xi^p = \eta^p = \zeta^p = 1$ and $\zeta = [\xi, \eta]$. Here $\{\zeta\} = \mathfrak{z}G = G'$, and G is the split extension of the elementary group $\{\xi, \zeta\}$ by $\{\eta\}$, with $\eta^{-1}\xi\eta = \xi\zeta$ and of course $\eta^{-1}\zeta\eta = \zeta$. We shall denote these two groups by

$$P^*_{p^3} \quad \text{and} \quad P_{p^3},$$

respectively. We state

<u>Lemma 14·35</u> For given p, there are exactly two ~~non-isomorphic~~ types of non-Abelian groups G of order p^3. ~~Groups~~ For both of them, $G' = \mathfrak{z}G$ is of order p and G/G' is elementary. When $p=2$, they are the octic and quaternion groups. When p is odd, they are the groups $P^*_{p^3} = P^*$ and $P_{p^3} = P$. ~~The~~ In P, $p+1$ subgroups of index p are all elementary, while in P^* there is a characteristic elementary subgroup $\Omega_1 P$ of order p^2 and the remaining p subgroups of index p are all cyclic.

It is easy to see that

$$|\operatorname{Aut} P| = (p^3 - p)(p^3 - p^2) \quad \text{and} \quad |\operatorname{Aut} P^*| = p(p^3 - p^2).$$

(F) To form a central product of two groups H and K, we have to establish an isomorphism identifying a subgroup Z of the centre of H with a subgroup of the centre of K. If $G = HK$ is the central product determined in this way, then $|G| = |H| \cdot |K| / |Z|$. The choice $Z = 1$ is always possible and then G is simply the direct product of H and K. However in speaking of a central product it will always be tacitly assumed that the amalgamated subgroup Z is $\neq 1$. If $\mathfrak{z}H$ and $\mathfrak{z}K$ are of order p a prime, this leaves only one possible choice $Z = \mathfrak{z}H = \mathfrak{z}K$. But $|\operatorname{Aut} Z| = p-1$ so that there are $p-1$ possible choices for the identifying isomorphism ~~automorphism~~). Nevertheless, if ~~[illegible]~~ the automizer of $Z = \mathfrak{z}H$ in Aut H is the full group Aut Z, then the central product $G = H\overset{K}{Z}$ will be determined to within isomorphism. For by definition $G = \overline{G}/Z_\theta$, where $\overline{G} = H \times K$ is the Cartesian product and Z_θ consists of all pairs (ζ, ζ^θ) with $\zeta \in Z$, and where θ is an isomorphism of $Z = \mathfrak{z}H$ onto $\mathfrak{z}K$. Under the assumption mentioned, the $p-1$ subgroups Z_θ of $\overline{G}$ are all conjugate under Aut $\overline{G}$.

When H is one of the non-Abelian groups of order p^3, then this assumption is in fact correct, as is easy to see. Hence there is no ambiguity in the definition of the central product of any number of such groups.

<u>Theorem 14·~~[illegible]~~4</u> (i) Let G be a p-group such that $G' = \mathfrak{z}G$ is of order p. Then $|G| = p^{2r+1}$ for some $r = 1, 2, \cdots$; G/G' is elementary; and G is expressible (in many ways if $r > 1$) as the central product $G_1 G_2 \cdots G_r$ of r non-Abelian groups of order p^3.

(ii) When $p = 2$, we have the central product isomorphisms

$$Q^r \cong O^2 Q^{r-2} \cong O^4 Q^{r-4} \cong \cdots$$

and $\quad OQ^{r-1} \cong O^3 Q^{r-3} \cong \cdots$

but Q^r and OQ^{r-1} are not isomorphic.

(iii) When p is odd, we have the central product isomorphisms

$$P^* P^{r-1} \cong (P^*)^2 P^{r-2} \cong \cdots \quad \cong (P^*)^r$$

but $P^* P^{r-1}$ and P^r are not isomorphic. Here $P = P_{p^3}$ and $P^* = P^*_{p^3}$.

Proof: From 7·1 (i), and $G' = \mathfrak{z}G$ of order p, it follows that G/G' is elementary viz. $\xi^p \in \mathfrak{z}G$ for all $\xi \in G$. In fact, in a group of class 2, the mapping $\xi \to [\xi, \alpha]$ is homomorphic for any fixed α.

Since G is not Abelian, we can choose ξ_1, η_1 so that $[\xi_1, \eta_1] \neq 1$. Then $G_1 = \{\xi_1, \eta_1\}$ is non-Abelian of order p^3. Let $\overline{G}_1 = C_G(G_1)$. Then $\overline{G}_1 = X_1 \cap Y_1$ where $X_1 = C_G(\xi_1)$ and $Y_1 = C_G(\eta_1)$. But ξ_1 and η_1 have each exactly p conjugates in G. Hence $|G : X_1| = |G : Y_1| = p$. Since ξ_1 and η_1 do not commute, $X_1 \neq Y_1$ and so $|G : \overline{G}_1| = p^2$. Also $G_1 \cap \overline{G}_1 = \mathfrak{z}G_1 = G'$ and so the elementary group G/G' is the direct product of G_1/G' and $\overline{G}_1/G'$. ~~Since~~ Hence $G = G_1 \overline{G}_1$ is a central product of G_1 and $\overline{G}_1$. Since $\mathfrak{z}G = G'$, we have $\mathfrak{z}\overline{G}_1 = G'$ also. Hence $\mathfrak{z}\overline{G}_1 = \overline{G}_1'$ is of order p and (i) follows by induction on $|G|$.

(ii) It will be sufficient to prove $Q^2 \cong O^2$. Let $G = Q^2$. Then $G = \{\xi_1, \eta_1, \xi_2, \eta_2\}$ and $G' = \{\zeta\}$, where $\zeta^2 = 1$ and $\xi_i^2 = \eta_i^2 = [\xi_i, \eta_i] = \zeta$ $(i = 1, 2)$, while $[\xi_1, \xi_2] = [\xi_1, \eta_2] = [\xi_2, \eta_1] = [\eta_1, \eta_2] = 1$. Here $G = G_1 G_2$ where $G_1 = \{\xi_1, \eta_1\}$ and $G_2 = \{\xi_2, \eta_2\}$ are two quaternion groups and $[G_1, G_2] = 1$. Let $G_1^* = \{\xi_1, \eta_1\eta_2\}$ and $G_2^* = \{\xi_1\xi_2, \eta_2\}$. Then $G = G_1^* G_2^*$ and $[G_1^*, G_2^*] = 1$ and here G_1^*, G_2^* are octic groups.

The group Q^2 contains 6 cyclic subgroups of order 4, three in G_1 and three in G_2, while OQ contains 10. More generally, Q^r contains $3r + 3^3\binom{r}{3} + 3^5\binom{r}{5} + \cdots = \frac{1}{2}\{(1+3)^r - (1-3)^r\} = 2^{r-1}\{2^r - (-1)^r\}$ cyclic subgroups of order 4; while OQ^{r-1} contains $2^{2r-2} + 2^{r-1}\{2^{r-1} + (-1)^r\} = 2^{r-1}\{2^r + (-1)^r\}$ cyclic subgroups of order 4. So Q^r and OQ^{r-1} cannot be isomorphic.

(iii) Let p be odd. If G is of exponent p, then all the central factors G_i must be of type $P = P_{p^3}$ and so $G = P^r$. It is sufficient to prove $(P^*)^2 \cong P^*P$. Let $G = G_1 G_2$ where $G_i = \{\xi_i, \eta_i\}$ and $[G_1, G_2] = 1$ and $\xi_i^p = [\xi_i, \eta_i]$ generates G_i', while $\eta_i^p = 1$, $i = 1, 2$. So G is of type $(P^*)^2$. We may assume that $\xi_1^p = \xi_2^{-p}$, so that $\xi_1\xi_2$ is of order p. Then $G_1^* = \{\xi_1\xi_2, \eta_2\}$ is of type P and if G_2^* is its centralizer, G is the central product $G_1^* G_2^*$. Here G_2^* is necessarily of type P^*, for otherwise G would be of exponent p, by 14.24.

If $G = G_1 G_2 \cdots G_r$ is of type P^*P^{r-1} with $G_2, \ldots, G_r$ of type P and $G_1 = \{\xi_1, \eta_1\}$ of type P^*, where $\eta_1^p = 1$, then $\Omega_1 G = \{\eta_1, G_2, \ldots, G_r\}$ is of index p. It is the direct product of $\{\eta_1\}$ with the group $G_2 \cdots G_r$ of type P^{r-1}.

<u>Lemma 14.41</u> If G is a p-group such that $G' = \mathfrak{z}G$ is of order p, then every automorphism of G which transforms G/G' identically is an inner automorphism.

Proof: Let $|G : G'| = p^{2r}$. Since $|G'| = p$, the number of automorphisms of G transforming G/G' identically is at most p^{2r}. This is also equal to $|G : \mathfrak{z}G|$, the number of inner automorphisms of G and the latter all transform G/G' identically. Hence the result.

If $C = H$, it is an easy consequence of Theorem 14.3 that $p = 2$. Then Theorem 14.3 implies that G is one of O_{2^n}, P_{2^n}, Q_{2^n}.

Suppose $C \supset H$ and p is odd. Since G/C is cyclic, Theorem 14.3 implies that $C = G$. Let $G_1 = \Omega_1(G).H$. By Lemma 14.41, $G = G_1C(G_1)$ and so $G = G_1$. By Lemma 14.4, we have $G = P^rH$, and we are done.

Finally, suppose $C \supset H$ and $p = 2$. If $|H| = 2$, the ~~lemma~~ theorem follows from Theorem 14.1, so suppose $|H| > 2$. Since C' is of order 2, it follows that $\Omega_1(C) = O_8^s\, Q_8^{s'} Z_1$, where $Z_1 = \Omega_2(H)$. Thus, C is the central product of $O_8^s\, Q_8^{s'}$ and H. If $C = G$, the proof is complete. Otherwise, we get $G = \Omega_1(C).C(\Omega_1(C))$ by Lemma 14.41. Furthermore, H is a maximal normal abelian subgroup of $C(\Omega_1(C))$. It follows that $C(\Omega_1(C))$ is one of P_{2^k}, Q_{2^k}, O_{2^k}. The various isomorphisms follow easily from Theorem 14.4.

(G) Let H be a p-group with $\mathfrak{z}H$ of order p and let Z be cyclic of order p^m, $Z = \{\zeta\}$. Then the central product HZ is defined to within isomorphism, because the identified subgroup can only be $Z_1 = \mathfrak{z}H$ with $Z_1 = \{\zeta^{p^{m-1}}\}$ and the automizer of Z_1 in $\mathrm{Aut}\, Z$ is $\mathrm{Aut}\, Z_1$.

<u>Theorem 14·5</u> Let G be a non-Abelian p-group such that every characteristic Abelian subgroup of G is cyclic and let $Z = \{\zeta\} = \mathfrak{z}G$ be of order p^m. Then

(i) for odd p, G is the central product ZP^r for some $r = 1, 2, \ldots$ where $P = P_{p^3}$ and the factor Z may be suppressed if $m = 1$. $|G| = p^{m+2r}$.

(ii) If $p = 2$ and $m = 1$, G is a central product of one of the forms Q^r, OQ^{r-1} $(r > 0)$ or $O_{2^k}Q^r$, $P_{2^k}Q^r$, $Q_{2^k}Q^r$ $(k > 3, r \geqslant 0)$ and no two of these are isomorphic. We have the central product isomorphisms

$$O_{2^k}Q^r \cong Q_{2^k}OQ^{r-1} \cong O_{2^k}O^2Q^{r-2} \cong Q_{2^k}O^3Q^{r-3} \cong \cdots$$
$$P_{2^k}Q^r \cong P_{2^k}OQ^{r-1} \cong P_{2^k}O^2Q^{r-2} \cong \cdots$$
$$Q_{2^k}Q^r \cong O_{2^k}OQ^{r-1} \cong Q_{2^k}O^2Q^{r-2} \cong O_{2^k}O^3Q^{r-3} \cong \cdots.$$

(iii) If $p = 2$ and $m > 1$, G is the central product ZQ^r for some $r > 0$ and we have the central product isomorphisms $ZQ^r \cong ZOQ^{r-1} \cong ZO^2Q^{r-2} \cong \cdots$

We need first

<u>Lemma 14·51</u> Let G be a p-group such that $\mathfrak{z}G'$ is cyclic. Then G' is cyclic.

Proof: Let $Z = \mathfrak{z}G'$. Since $Z \triangleleft G$, $Z \leq G'$, we can choose a normal cyclic subgroup H of G which is maximal subject to $Z \leq H \leq G'$. If G' is not cyclic, then $H < G'$ and by 5·2 (i), there is a normal subgroup K of G such that $H < K \leq G'$ and $|K : H| = p$. Then K is not cyclic. If K is Abelian, it is of type $(m, 1)$ where $|H| = p^m$, and $\Omega_1 K$ is of order p^2. If K is not Abelian, $\Omega_1 K$ is still of order p^2 for odd p, by 14·3 (i); and also for $p = 2$ provided K is of class 2, by 14·3 (iv). The only alternative is for K to be one of the groups $O_{2^{m+1}}$, $P_{2^{m+1}}$, $Q_{2^{m+1}}$ by 14·3 (iv) and (v). But K contains a subgroup L of order p^2 and normal in G. Hence $L \leq \mathfrak{z}^2 G$ and so $[L, G'] = 1$ by 7·8 (ii). Hence $L \leq \mathfrak{z}K$. But the groups $O_{2^{m+1}}$, ... have centre of order 2, so K cannot be one of these. We conclude that K has a characteristic subgroup $M = \Omega_1 K$ of order p^2. Then $M \triangleleft G$ and so, as we have just seen, $[M, G'] = 1$. This contradicts the assumption that $\mathfrak{z}G'$ is cyclic.

Proof of 14.5. Since $\mathfrak{z}G'$ is a characteristic Abelian subgroup of G, it is cyclic, by hypothesis. Hence G' is cyclic by 14.51, and ZG' is a characteristic Abelian subgroup of G. So ZG' is cyclic too. Let H be a maximal characteristic cyclic subgroup of G containing ZG'. Then ~~[illegible]~~ $H < G$, and if $C = C_G(H)$, then $\mathfrak{z}C$ is a characteristic Abelian subgroup of G containing H., hence $\mathfrak{z}C$ is cyclic, and so $\mathfrak{z}C = H$. Also $C' \leq G' \leq H = \mathfrak{z}C$. If $C > H$, then C is of class 2 and $[\xi^r, \eta^s] = [\xi, \eta]^{rs}$ for all ξ, η in C, by 7·1 (i), (ii). Let the Abelian group C/H be of type $(r_1, r_2, \ldots)$, where $r_1 \geqslant r_2 \geqslant \cdots$, and let $H\xi_1, H\xi_2, \ldots$ be a basis of C/H. Then we have $[\xi_1^{p^{r_2}}, \xi_i] = [\xi_1, \xi_i^{p^{r_2}}] = 1$ for $i = 2, 3, \ldots$ and hence $\xi_1^{p^{r_2}} \in H = \mathfrak{z}C$. Hence $r_1 = r_2$. If $r_1 > 1$, we have $[\xi_i^{p^{r_1-1}}, \xi_j^{p^{r_1-1}}] = [\xi_i^{p^{2r_1-2}}, \xi_j] = 1$ for all i, j. If $L/H = \mho_{r_1-1}(C/H)$, it ~~follows~~ would follow that L char G, $L' = 1$, $L > H$, hence L cyclic, contrary to the choice of H. Hence $r_1 = 1$ and C/H is elementary, and so C' is of order p.