§10. Systemizers and Carter Subgroups of Soluble Groups

(A) Let G be a soluble group and let $p_1, \dots, p_k$ be the distinct primes which divide $|G|$. By 9·5, G has for each $i = 1, 2, \dots, k$ at least one $S_{p_i'}$-subgroup H_i. By 9·6 (iii), the 2^k intersections of sets of subgroups chosen from $H_1, \dots, H_k$, including the empty intersection G, are S_ϖ-subgroups of G for various ϖ. In particular $P_i = \bigcap_{j \neq i} H_j$ is a Sylow p_i-subgroup of G. Since for $i \neq l$

$$|P_i P_l| = |\bigcap_{j \neq i, l} H_j| = |P_l P_i|,$$

we must have $P_i P_l = \bigcap_{j \neq i, l} H_j = P_l P_i$, so that the Sylow subgroups $P_1, P_2, \dots, P_k$ of G are permutable in pairs. And if Λ is any subset of $1, 2, \dots, k$ and Λ' is the complementary set, we have more generally

$$\prod_{i \in \Lambda} P_i = \bigcap_{j \in \Lambda'} H_j .$$

Thus the 2^k intersections of sets of H_j coincide with the 2^k products of sets of P_i, including the empty product 1. This set of 2^k subgroups is called an <u>S-system</u> of G. Given any S-system $\underset{\sim}{T}$ of G, and any set of primes ϖ, just one member of $\underset{\sim}{T}$ is an S_ϖ-subgroup of G and this will be denoted by T_ϖ. If ϖ and ψ have the same intersection with the set $p_1, \dots, p_k$, then $T_\varpi = T_\psi$. $\underset{\sim}{T}$ is determined uniquely by the T_p and equally well by the $T_{p'}$. For any two sets of primes ϖ and ψ, we have

$$T_\varpi T_\psi = T_\psi T_\varpi = T_{\varpi \cup \psi}$$

and

$$T_\varpi \cap T_\psi = T_{\varpi \cap \psi} .$$

In particular, $T_\varpi T_{\varpi'} = G$ and $T_\varpi \cap T_{\varpi'} = 1$.

<u>Lemma 10·1</u> Let $\underset{\sim}{S}$ and $\underset{\sim}{T}$ be any two S-systems of the soluble group G. Then there is an element $\xi \in G$ such that $S_\varpi^{\xi} = T_\varpi$ for all ϖ.

Proof: Suppose that $S_{p_i'} = T_{p_i'}$ for $i = 1, 2, \dots, r-1$ $(0 \leq r \leq k)$ while $S_{p_r'} \neq T_{p_r'}$. Since $S_{p_r'}$ and $T_{p_r'}$ are conjugate in G by 9·5 and $S_{p_r'} S_{p_r} = G$, there is an element $\eta \in S_{p_r}$ such that $S_{p_r'}^{\eta} = T_{p_r'}$. But

$S_{p_r} \le S_{p_i'}$ for $i = 1, 2, \dots, r-1$. Hence the S-system $\underset{\sim}{S}^{\eta}$ of G coincides with $\underset{\sim}{T}$ in its first r Sylow complements $T_{p_1'}, \dots, T_{p_r'}$. By induction on r, $\underset{\sim}{S}^{\xi} = \underset{\sim}{T}$ for some $\xi \in G$.

(B). The normalizers in G of the S-systems $\underset{\sim}{T}$ of G will be called the <u>systemizers</u> of G. By definition,

$$N_G(\underset{\sim}{T}) = \bigcap_{\varpi} N_G(T_\varpi) = \bigcap_{p} N_G(T_p) = \bigcap_{p} N_G(T_{p'}).$$

If H is any subgroup of G and $\underset{\sim}{V}$ is an S-system of H, then

$$N_G(\underset{\sim}{V}) = \bigcap_{\varpi} N_G(V_\varpi) = \bigcap_{p} N_G(V_p) = \bigcap_{p} N_G(V_{p'})$$

will be called a <u>relative systemizer</u> of H in G. Since $H = \prod_p V_p$, it is clear that $N_G(\underset{\sim}{V}) = N_L(\underset{\sim}{V})$, where $L = N_G(H)$. $H \cap N_G(\underset{\sim}{V}) = N_H(\underset{\sim}{V})$ is a syste

<u>Lemma 10.2</u> Let G be a soluble group and let $H \lhd G$.

(i) The relative systemizers L of H in G are all conjugate in G and $LH = G$. In particular any two systemizers of G are conjugate in G.

(ii) The systemizers of G are nilpotent.

Proof: (i) Let $L = N_G(\underset{\sim}{V})$ and $M = N_G(\underset{\sim}{W})$ where $\underset{\sim}{V}$ and $\underset{\sim}{W}$ are S-syste of H. By 10.1, $\underset{\sim}{W} = \underset{\sim}{V}^{\xi}$ for some $\xi \in H$. Hence $M = L^{\xi}$. If $\eta \in G$ then $\underset{\sim}{V}^{\eta}$ is an S-system of H since $H \lhd G$. Hence $\underset{\sim}{V}^{\eta\zeta} = \underset{\sim}{V}$ for some $\zeta \in$ by 10.1 and so $\eta\zeta \in L$, $\eta \in LH$. This is true for all $\eta \in G$ so $G = L$

(ii) Take $H = L$ and let P be a Sylow p-subgroup of L. Then P normalizes V_p which is now a Sylow p-subgroup of G. Hence $P \le V_p$, $P = L \cap V_p$ and so $P \lhd L$ since L normalizes V_p. Thus L is nilpotent.

(C) Now let M be a minimal normal subgroup of the soluble group G so that $|M| = q^m$ for some prime q; and let $\underset{\sim}{T}$ be an S-system of G and $H = N_G(\underset{\sim}{T})$ the corresponding systemizer. The groups MT_ϖ/M are permutable in pairs and by 5.7, MT_ϖ/M is an S_ϖ-subgroup of G/M. Since $MT_\varpi = \prod_{p \in \varpi} MT_p$, the groups $T_\varpi^* = MT_\varpi/M$ form an S-system $\underset{\sim}{T}^*$ of G/M. Let $H^*/M = N_{G/M}(\underset{\sim}{T}^*)$. Thus $H^* = \bigcap_p N_G(MT_{p'})$, and so $HM \le H^*$. Conversely let $\xi \in H^*$. If $p \ne q$, then $MT_{p'} = T_{p'}$. Also $T_{q'}^{\xi}$ is an $S_{q'}$-subgroup of $MT_{q'}$ and hence it is conjugate to $T_{q'}$ in $MT_{q'} = T_{q'}M$. Thus $T_{q'}^{\xi} = T_{q'}^{\eta}$ for some $\eta \in M$. Hence $\eta \in T$

for all $p \neq q$ and so $\xi\eta^{-1} \in H$, $\xi \in HM$. Thus $H = H^*$.

By repeated application of this argument we obtain

~~Theorem~~ Lemma 10.3 Let G be a soluble group, $K \lhd G$ and let $\underset{\sim}{T}$ be an S-system of G, $T^*_\varpi = KT_\varpi/K$. The the set $\underset{\sim}{T}^*$ of groups T^*_ϖ is an S-system of G/K; and if $L = N_G(\underset{\sim}{T})$, $L^* = N_{G/K}(T^*)$, then $L^* = KL/K$.

In other words, in any epimorphism of a soluble group G onto another group G^*, the S-systems of G map into the S-systems of G^* and the systemizers of G map into the systemizers of G^*.

Now let M be once more a minimal normal subgroup of G with $|M| = q^m$, q prime; and let $C = C_G(M)$. If $C = G$, so that $M \leq \mathfrak{z}G$ it is clear that every systemizer of G contains M. If $C < G$, let D/C be a chief factor of G. Then by 9.3 (ii), D/C is a q'-group. ~~...~~ If $\underset{\sim}{T}$ is any S-system of G and $L = D \cap T_{q'}$, it follows that $LC = D$ by 5.7. Let $H = N_G(\underset{\sim}{T})$ and $N = H \cap M$. Then N normalizes L and we have $[L, N] \leq L \cap M = 1$. Hence L centralizes N. But C also centralizes N. Hence $N \leq \mathfrak{z}D \lhd G$. If $N \neq 1$, it would follows that $M \leq \mathfrak{z}D$ since M is a minimal normal subgroup of G, contrary to the definition of $C = C_G(M)$. Hence $N = 1$.

Using 10.3, this gives

<u>Theorem 10.4</u> Let H/K be any chief factor of the soluble group G and let L be any systemizer of G. Then $H \leq LK$ or $H \cap L \leq K$ according as H/K is a central factor of G or not.

We may express this result by saying that <u>the systemizers of G cover every central chief factor of G and avoid every non-central chief factor of G</u>. Now the index $|G : L|$ of a systemizer of G in G is equal to the number of distinct S-systems of G. Hence we have

<u>Corollary 10.41</u> Let $G = G_0 > G_1 > \cdots > G_n = 1$ be a chief series of the soluble group G. Then the number of distinct S-systems of G is equal to $\prod |G_{i-1} : G_i|$ taken over all the non-central factors G_{i-1}/G_i of the given chief series.

Subgroups and direct products of nilpotent groups are nilpotent. By 8·5, it follows that every group G has a uniquely determined normal subgroup

$$G^{\mathfrak{N}} = \gamma_\infty G,$$

the limit of the lower central series of G, with the property that a quotient group G/K of G is nilpotent if and only if $K \geqslant G^{\mathfrak{N}}$.

Corollary 10·42 If L is a systemizer of the soluble group G, then $G = LG^{\mathfrak{N}} = \{L^\xi ; \xi \in G\}$.

For every chief factor H/K of G with $K \geqslant G^{\mathfrak{N}}$ is a central factor of G and so $H \leqslant LK$ by 10·4. Hence $G = LG^{\mathfrak{N}}$. If M is a maxima[l] normal subgroup of G, then G/M is cyclic of prime order and so $M \geqslant G^{\mathfrak{N}}$. It follows that $L \not\leqslant M$. Hence G is the normal closure of L in G.

(C) We consider next the relations between the systemizers of a soluble group G and the abnormal maximal subgroups of G. First we need

Lemma 10·43 Let $\underset{\sim}{T}$ be an S-system of the soluble group G, let $H = N_G(\underset{\sim}{T})$ and let $N = N_G(T_{p'})$. Then the Sylow p-subgroup P of H is a Sylow p-subgroup of N and $|P|$ is equal to the product of the orders of the central p-factors in any chief series of G.

Proof: Let P^* be any Sylow p-subgroup of N. For $q \neq p$, there is an $S_{q'}$-subgroup $T_{q'}^*$ of G which contains P^*, by 9·5. Let $T_{p'}^* = T_{p'}$ and let $\underset{\sim}{T}^*$ be the ~~Sy~~ S-system of G determined by the $T_{p'}^*$. Then $P^* \leqslant H^* = N_G(\underset{\sim}{T}^*) = \bigcap_q N_G(T_{q'}^*)$ so $|N|_p = |P^*| \leqslant |H^*|_p = |H|_p = |P|$ and $|P| \leqslant |N|_p$ since $P \leqslant N$. Hence $|P| = |P^*|$.

The second part of 10·43 ~~now~~ follows from 10·4.

Now let M be an abnormal maximal subgroup of the soluble group G and suppose first that $K_G(M) = \bigcap_{\xi \in G} M^\xi = 1$. By 9·2, G has a unique minimal normal subgroup $L = \mathfrak{N}G$ of order p^m, p prime, and $M \cap L = 1$, $ML = G$. Hence M contains an $S_{p'}$-subgroup $T_{p'}$ of G. If $q \neq p$ and $T_{q'}$ is any $S_{q'}$-subgroup of G, then $L \leqslant T_{q'}$ and so $M \cap T_{q'} = S_{q'}$ is an $S_{q'}$-subgroup of M. Taking $S_{p'} = T_{p'}$, we thus obtain an S-syste[m]

$\underset{\sim}{S}$ of M and an S-system $\underset{\sim}{T}$ of G. Since M is not normal in G, we have $M=$ and so $L/1$ is a non-central chief factor of G by 9·2. Since M and G/L incident sections of G, it follows that the product of the orders of the central p-factors in a chief series is the same for M as for G. Hence $N_G(T_{p'})$ and $N_M(T_{p'})$ have Sylow p-subgroups of the same order. Therefore $N_G(T_{p'})=N_M(T_{p'})$ and $H=N_G(\underset{\sim}{T}) \leq M$.

In the general case where $K=K_G(M) \neq 1$, we can apply this result to the group G/K. Let $\underset{\sim}{S}$ and $\underset{\sim}{T}$ be S-systems of M and G respectively, such that $S_{p'}=T_{p'}$. Here as before we assume $|G:M|=p^m$. For $q \neq p$ we have $|T_{q'}:S_{q'}|=p^m$ and so $MT_{q'}=G$, $M \cap T_{q'}=S_{q'}$. Since $N_G(T_{p'}) \leq N_G(KT_{p'})=N_M(KT_{p'})$ by the above particular case, the result $N_G(T_{p'})=N_M(T_{p'})$ holds in general. Hence again $H=N_G(\underset{\sim}{T}) \leq M$ and therefore $H \leq N_M(\underset{\sim}{S})$.

A subgroup H of a group G is called <u>subabnormal</u> in G if there is a chain of subgroups H_i such that

$$H=H_1 \text{ abn } H_2 \text{ abn } \cdots H_{r-1} \text{ abn } H_r = G.$$

Since every subgroup containing an abnormal subgroup is abnormal, we may also say that H is subabnormal in G if and only if H can be linked to G by a chain of subgroups each of which is maximal but not normal in the next.

We can now state

<u>Theorem 10·5</u>. Let G be a soluble group and let L be a subabnormal subgroup of G. Then

(i) every systemizer of L contains a systemizer of G.

(ii) If $|G:L|=p^m$, p prime, then L contains the normalizer in G of some $S_{p'}$-subgroup of G and L is abnormal in G.

(iii) The systemizers of G are the minimal subabnormal subgroups of G.

Proof: (i) $L=L_0<L_1<\cdots<L_r=G$ where each L_{i-1} is an abnormal maximal subgroup of L_i. As we have seen, this implies that every systemizer of L_{i-1} contains a systemizer of L_i.

(ii) In this case $|L_i : L_{i-1}| = p^{m_i}$ say and if T is any $S_{p'}$-subgroup of L, then T is an $S_{p'}$-subgroup of G and we have $N_{L_{i-1}}(T) = N_{L_i}(T)$ for each i since L_{i-1} is abnormal and maximal in L_i. Hence $N_L(T) = N_G(T)$. But $N_G(T)$ is abnormal in G, since T pron G by 9·5. Hence L is abnormal in G.

(iii) By (i), we have only to show that $H = N_G(\underset{\sim}{S})$ is subabnormal in G, where $\underset{\sim}{S}$ is any S-system of G. Let $G = G_0 > G_1 > \cdots > G_m = 1$ be a chief series of G and let $H_i = N_G(\underset{\sim}{S}^{(i)})$, where $S_{\varpi}^{(i)} = S_{\varpi} \cap G_i$, so that $\underset{\sim}{S}^{(i)}$ is an S-system of G_i by 5·7 and H_i is the corresponding relative systemizer ...
Clearly $H = H_0 \leq H_1 \leq \cdots \leq H_m = G$. Now $|G_{i-1} : G_i| = p^n$ for some prime $p = p(i)$ and some $n = n(i)$. Hence $S_q^{(i)} = S_q^{(i-1)}$ for $q \neq p$, and so

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p(i).

H_{i-1} is the normalizer of $S_p^{(i)}$ in H_{i}. Now $H_i \cap G_i = N_{G_i}(\underset{\sim}{S}^{(i)})$ is a systemizer of G_i and so $H_i \cap S_p^{(i)} = P_i$ is the Sylow p-subgroup of the nilpotent group $H_i \cap G_i$. Hence $H_{i-1} \leq N_{H_i}(P_{i-1})$. But $S_p^{(i-1)} = S_p^{(i)}$ since G_{i-1}/G_i is a p-group and $H_i G_i \geqslant G_{i-1}$. Hence every element of $N_{H_i}(P_{i-1})$ normalizes $S_p^{(i-1)}$, and also $S_q^{(i-1)} = S_q^{(i)}$ for all $p \neq q$. It follows that $N_{H_i}(P_{i-1}) \leq H_{i-1}$. Consequently $H_{i-1} = N_{H_i}(P_{i-1})$. delete: see below [Now P_{i-1} is a Sylow subgroup of the normal subgroup $H_i \cap G_{i-1}$ of H_i, so that P_{i-1} pron H_i] It follows that H_{i-1} abn H_i and, since this is true for each i, H is subabnormal in G.

The argument in (iii) shows rather more viz.

<u>Lemma 10·51</u> Let $K \triangleleft G$, let K be soluble and let $\underset{\sim}{T}$ be any S-system of K. Then the relative systemizer $N_G(\underset{\sim}{T})$ is subabnormal in G.

Note that although systemizers are subabnormal subgroups, not every subgroup which contains a systemizer of the soluble group G need be subabnormal in G. For example, if G is the octahedral group and if H is one of the three elementary non-normal subgroups of order 4, then H contains two of the six systemizers of G (which are of order 2) but H is not subabnormal in G since the only proper subgroup of G which contains H is one of the Sylow 2-subgroups S and $H \triangleleft S$. However H pron G.

~~delete and replace by:~~ ~~since P_{i-1} is the S_p-subgroup of H_{i-1},~~

~~$P(H_i \cap S^{(i)}) \; H_i \cap S^{(i-1)}$~~

H_{i-1} is the normalizer of $S_p^{(i-1)}$ in H_i. Since $G_{i-1} = G_i(S_p^{(i-1)} \cap H_i)$, it follows that $|S_p^{(i-1)} \cap H_i| = |G_{i-1} : G_i| \, |S_p^{(i)} \cap H_i|$. If P^* is a Sylow p-subgroup of $G_{i-1} \cap H_i$, then $P^* = |G_{i-1} : G_i| \, |P^* \cap G_i|$. Since $P^* \cap G_i$ is in a systemizer of G_i, it follows that $|P^*| \leq |S_p^{(i-1)} \cap H_i|$. Thus, $S_p^{(i-1)} \cap H_i$ is a Sylow p-subgroup of $G_{i-1} \cap H_i$. Since $S_p^{(i-1)} = S_p^{(i)}(S_p^{(i-1)} \cap H_i)$, it follows that $H_{i-1} = N_{H_i}(S_p^{(i-1)} \cap H_i)$. Since $S_p^{(i-1)} \cap H_i$ is a Sylow p-subgroup of $G_{i-1} \cap H_i \lhd H_i$, it follows that $S_p^{(i-1)} \cap H_i$ prn H_i, so H_{i-1} abn H_i.

(D) A group G is called <u>metanilpotent</u> if it has a normal subgroup K such that both K and G/K are nilpotent. For such a K, clearly $G^{\mathfrak{N}} \leq K \leq \mathfrak{N}G$. G is metanilpotent if and only if $G/\mathfrak{N}G$ is nilpotent or equivalently if and only if $G^{\mathfrak{N}}$ is nilpotent.

<u>Theorem 10.6</u> Let G be metanilpotent and let H be a systemizer of G. Then H is a systemizer of every subgroup L of G which contains it. Also H is abnormal in G.

Proof: Let $F = \mathfrak{N}G$ be the Fitting subgroup of G. Since G is metanil[potent,] G/F is nilpotent and so $HF = G$ by 10.42. The group $L \cap F$ is nilpot[ent] and so also is H by 10.2 (ii). Let R_p and S_p be the Sylow p-subgrou[ps] of $L \cap F$ and H respectively. Then R_p char $L \cap F \lhd L$ and so $R_p \lhd L$. Hence $T_p = R_p S_p$ is a p-subgroup of L. Since $HF = G$, we have $H(L \cap F) = L$ and so T_p is a Sylow p-subgroup of L. Since $R_p \lhd L$ and $S_p \lhd H$, we have $T_p T_q = T_q T_p$ for any two primes p and q. Hence the groups T_p and their products form an S-system $\underset{\sim}{T}$ of L and $H \leq H^* = N_L(\underset{\sim}{T})$.

Now let C/D be any chief factor of G such that $C \leq F$. By 9.3 (i), F centralizes C/D. Let $N = N_G(H)$. If $H < N$, there exists such a chief factor C/D with $H \cap C \leq D$ and $D(N \cap C) > D$, by 9.4. Then $[H, N \cap C] \leq H \cap C \leq D$ and since $HF = G$, it follows that $[G, N \cap C] \leq D$. Hence $D(N \cap C) \lhd G$ and consequently $C = D(N \cap C)$ and C/D is a central factor of G. But in that case $C \leq HD$ by 9.4, contrary to $H \cap C \leq D$. This contradiction shows that $H = N$ is disnormal in G. It now follows that $H = H^*$ is a systemizer of L, since H^* is nilpotent by 10.2 (ii).

Finally, let $H \leq L \cap L^{\xi}$, where $\xi \in G$. Then $\xi H \xi^{-1}$ and H are two systemizers of L and hence they are conjugate in L. Hence there is an element η of L such that $H^{\xi^{-1}\eta} = H$. Then $\xi^{-1}\eta \in N_G(H) = H$ and so $\xi \in L$. Since L is any subgroup of G containing H, it follows that H is abnormal in G.

This theorem like the next is due to R. Carter.

In any group G, a nilpotent subgroup which is dianormal in G will be called a <u>Carter subgroup</u> of G.

<u>Theorem 10.7</u> Let G be a soluble group. Then

(i) G contains at least one Carter subgroup.

(ii) The Carter subgroups of G are all conjugate in G.

(iii) If H is a Carter subgroup of G and if L is any subgroup of G containing H, then H is a Carter subgroup of L.

(iv) The Carter subgroups of G are abnormal in G.

(v) If $K \lhd G$, and if H is a Carter subgroup of G, then KH/K is a Carter subgroup of G/K.

(vi) The Carter subgroups of G coincide with the systemizers of G if and only if the latter are dianormal in G, and this is the case in particular when G is metanilpotent.

Proof by induction on $|G|$. ~~First let G be metanilpotent. Then by 10 the systemizers of G are Carter subgroups. By~~ If G is nilpotent, then the only Carter subgroup of G is G itself and the results (i)–(vi) are immedia[te]. Hence we suppose that G is not nilpotent. Let M be a minimal normal subgroup of G, $|M| = p^m$.

(i) By induction G/M has a Carter subgroup L/M. Since L/M is nilpotent, L is metanilpotent. Let H be a systemizer of L and let $\xi \in N_G(H)$. Since $M \lhd G$ and $HM = L$, we have $\xi \in N_G(L) = L$ and so $\xi \in N_L(H) = H$ by 10.6. But H is nilpotent. Hence H is a Carter subgroup of G.

(ii) Let H^* be any Carter subgroup of G and let $L^* = H^*M$. Then $L^*/M \cong H^*/M \cap H^*$ is nilpotent and so L^* is metanilpotent. If $L^* < G$, it follows from (ii) by induction that H^* is a systemizer of L^*, since H^* is a Carter subgroup of L^* and the systemizers of metanilpotent groups are also Carter subgroups. If $L^* = G$, then $H^* \cap M \lhd G$ and so is either 1 or M. But G is not nilpotent. Hence $H^* \cap M = 1$. H^* normalizes

$MS_{q'} = T_{q'}$ where $S_{q'}$ is the unique $S_{q'}$-subgroup of H^* for each prime $q \neq p$ and H^* also normalizes $S_{p'} = T_{p'}$. Since $T_{q'}$ is an $S_{q'}$-subgroup of $G = H^*M$ it follows that $H^* \leq N_G(\underset{\sim}{T})$ where $\underset{\sim}{T}$ is the S-system of G determined by the $T_{q'}$. But M is a non-central chief factor of G and so $N_G(\underset{\sim}{T}) \cap M = 1$ by 9.4. Since $G = H^*M$, it follows that $H^* = N_G(\underset{\sim}{T})$ is a systemizer of $G = L$ in this case also.

Let $\xi \in N_G(L^*)$. Then $H^{*\xi}$ is also a systemizer of L^*, hence conjugate to H^* in L^* and so $H^{*\xi\eta} = H^*$ for some $\eta \in L^*$. Therefore $\xi\eta \in N_G(H^*) = H^*$ and so $\xi \in L^*$. Thus $N_G(L^*) = L^*$ and since L^*/M is nilpotent it follows that L^*/M is a Carter subgroup of G/M. By induction, L^* is conjugate to L in G and hence H^* is conjugate to H.

(iii) is clear.

(iv) Let $H \leq L \cap L^{\xi}$, where $\xi \in G$ and L is a subgroup of G containing the Carter subgroup H of G. Then H and $\xi H \xi^{-1}$ are Carter subgroups of L, hence conjugate in L by (ii), so that $\xi^{-1}\eta \in N_G(H) = H$ for some $\eta \in L$. It follows that $\xi \in L$. Thus H is abnormal in G.

(v) This has been shown in proving (i) and (ii) for the case $K = M$, a minimal normal subgroup of G. The general result follows at once.

(vi) Since Carter subgroups are dianormal by definition and systemizers are nilpotent by 10.2 (ii), this is clear.

(E) Following Wielandt, we call a subgroup H of G <u>intravariant</u> in G if H^{α} is conjugate to H in G for every $\alpha \in \mathrm{Aut}\, G$. Equivalently H is intravariant in G if and only if the class of conjugates in G to which H belongs is a <u>characteristic class of conjugates in G</u> i.e. is invariant under $\mathrm{Aut}\, G$. A normal subgroup is intravariant if and only if is characteristic.

<u>Lemma 10.8</u> (i) If H is intravariant in K and K is intravariant in G then H is intravariant in G.

(ii) If H is intravariant in G, so also are $N_G(H)$ and $C_G(H)$.

(iii) If H is intravariant in K and $K \triangleleft G$, then $K N_G(H) = G$.

(iv) Sylow subgroups are intravariant.

(v) In a soluble group, S_{ϖ}-subgroups, systemizers and Carter subgroups are intravariant.

The proofs are immediate for (i), (ii), (iii). (iv) follows from Sylow's Theorem and (v) from 9.5, 10.1 and 10.7.